Problems of the student contest of the St. Petersburg Mathematical Society, 2010.

Please send solutions in .tex or .pdf to `fedyapetrov@gmail.com` .

1. The plane is partitioned into parabolas (each point belongs to exactly one parabola). Does it follow that their axes have the same direction?

2. a) Does there exist an infinite-dimensional separable Banach space X such that any Banach space Y isomorphic to X is linearly isometric to X?

   b) What about a non-separable space?

3. Consider the claim $P(k,n)$: if -1 in some field is a sum of k squares, then it is also a sum of finite number of n-th powers.

   a) Prove $P(1,n)$ for any n.

   b) Prove $P(k,4)$ for any k.

   c) Is $P(k,n)$ true for all k, n?

4. The function $f : [0,1] \to \mathbb{R}$ satisfies the equation $f(x) = f(\frac{x}{2}) + f(\frac{x+1}{2})$ for any $x \in [0,1]$. Does it follow that $f(x) = c(1-2x)$ for some real c if

   a) $f \in C^2[0,1]$; b) $f \in C^1[0,1]$; c) $f \in C[0,1]$?

5. Let $p > 3$ be a prime number and let a, b be integers such that p divides a^2+ab+b^2. Prove that $(a+b)^p - a^p - b^p$ is divisible by a) p^2; b) p^3.

6. Prove that arbitrary (not necessary countable) union of closed non-degenerated a) segments on the real line; b) triangles on the plane is Lebesgue measurable.

7. Prove that for any positive integer n there exist a polygon F_0 in the plane and its shifted copies $F_1, F_2, \ldots, F_n$ such that no two of the polygons F_i $(0 \leqslant i \leqslant n)$ have common interior point but F_0 and F_i have at least one common (boundary) point for each $i = 1, 2, \ldots, n$.

8. Let S be a sphere and P a point in $\mathbb{R}^d$. Define the characteristic of a simplex T inscribed in this sphere as

   a) the volume of T times the sum of squares of the distances from P to the vertices of T;

   b) the volume of T times the sum of the squares of mutual distances between the vertices of T.

   Given a polyhedron inscribed in S, prove that if it is partitioned into simplices inscribed in S, then the sum of their characteristics does not depend on the partitioning α) for $d = 2$; β) for $d = 3$; γ) for arbitrary d (thus, there are six questions in this problem, enumerated by the elements of the set $\{a, b\} \times \{\alpha, \beta, \gamma\}$).

9. Consider the topological space $\mathbb{R}^\infty$ of all real sequences $(x_1, x_2, \ldots)$ equipped with the product topology. Say that two sequences x, y are equivalent if $x = \lambda y$ for some $\lambda > 0$. Denote by S the space of nonzero equivalence classes equipped with the factor topology (S is, in a sense, an infinite-dimensional sphere). Prove that the space S is

   a) a Hausdorff space,

   but

   b) any continuous function $S \to \mathbb{R}$ is constant.

10. For $q > 1$, put

$$F(t) = \int\limits_{x_1(t)}^{x_2(t)} \frac{x\,dx}{\sqrt{1 + tx - |x|^q}},$$

where x_1 and x_2 are the roots of the denominator. Clearly, $x_{1,2}(0) = \mp 1$ and $F(0) = 0$.

a) Calculate $F'(0)$.

b) Find all values of q for which $F'(0) = 0$.

11. For a positive rational x denote by $\ell(x)$ the height of the continued fraction of x (in other words, $\ell(x)$ is the number of steps in the Euclid algorithm for the numerator and the denominator of a fraction x.) So, $\ell(17/5) = 3$, since $17/5 = 3 + 1/(2 + 1/2)$. Prove that a) for $r = 2$; b) for any positive rational r, there exist constants $c_1(r)$, $c_2(r)$ such that the inequality $c_1(r) \cdot \ell(x) \leqslant \ell(rx) \leqslant c_2(r) \cdot \ell(x)$ holds for any positive rational number x.

    Try to get the estimates for $c_1(r)$, $c_2(r)$ in terms of r as sharp as you can.

12. Let a graph G (non-directed, without loops and multiple edges) with $3n$ vertices be given. Assume that for any $n+1$ vertices of G there exist two of them which are not joined by an edge but the set of vertices of G can be partitioned into three parts so that any two vertices of the same part are joined. Denote by $f(n)$ the minimum number of colors sufficient to properly color such a graph. (Recall that the coloring is called proper if no edge connects the vertices of the same color).

    a) Prove that $f(5) = 8$.

    б) Prove that $f(n) \leqslant 8n/5$ for each n.