

A BY-PROXY TALK ON ALEXANDROV'S CONTRIBUTION

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ABSTRACT. This talk presents the unpublished articles by O. A. Ladyzhenskaya, Yu. G. Reshetnyak, and V. A. Zalgaller concerning the nomination of A. D. Alexandrov for the Wolf Prize in 1995.

FOREWORD

Since the Organizing Committee was unsure that Professor Ladyzhenskaya could make her talk today, Professor Burago asked me on last Friday to make a replacement. Of course, I have prepared a technical talk about some “soap” isoperimetric-type problems in convex geometry for our meeting but this special topic is hardly appropriate for the opening ceremony.

Also, I must confess that it is completely impossible for me to render anything as witty and beautiful as typical of Olga Aleksandrovna. Fortunately, there is an illuminating and rewarding theorem by Mikhail Bulgakov, a renowned Russian writer, claiming that “Manuscripts Don’t Burn” as stated in his celebrated book “The Master and Margaret.” As a consequence, I retain a few unknown and unpublished articles about the contributions of Alexandrov which were written and/or initiated by Professor Ladyzhenskaya in 1995.

These articles appeared in the course of nominating Alexandrov for the Wolf Prize (which he never won by some reasons still unknown to me completely). Olga Aleksandrovna asked me in the summer of 1995 to translate and polish the nominating texts and so they all have resided in my computer since then. I was and am still proud of my participation in this matter intended as a small tribute to our mutual teacher and long-term friend.

I hope that the genuine words of Professors Ladyzhenskaya, Reshetnyak, and Zalgaller about Alexandrov are the best choice for the opening ceremony of this meeting dedicated to the memory of *Alexandr Danilovich Alexandrov*, a geometrical giant of the twentieth century.

NOMINATION BY LADYZHENSKAYA

At the turn of this century,¹ geometry came to search into objects “in the large.” However, the methods of differential geometry, and all the more the methods of studying solvability of the Cauchy problem and boundary value problems for partial differential equations which were developed in the 19th century provided no approach to their resolution. The efforts of such outstanding mathematicians as Minkowski, Hilbert, H. Weyl, et al. yielded only fragmentary results. At the same time their works contained the statements of many important unresolved problems which predetermined the development of geometry “in the large” in this century.

Fundamental achievements in research into the problems belong to Alexandrov. They contain the resolving of many difficult specific problems as well as the propounding of a general theory:

(a) establishment of differential geometry on nonsmooth surfaces which is a far-reaching generalization of the classical differential geometry created by Gauss, Riemann, et al.;

A complete version of a talk at the Second Russian–German Geometry Meeting Dedicated to the 90th Anniversary of A. D. Alexandrov (1912–1999).

¹This was written in 1995.

(b) origination of direct methods for research into nonlinear problems in classes of convex surfaces and further in a more general class of “manifolds of bounded curvature.”

Constructions of a general character formed the contents of his monographs “Intrinsic Geometry of Convex Surfaces” [in Russian], GITTL, Moscow–Leningrad (1948), 378 p.² and “Two-Dimensional Manifolds of Bounded Curvature” [in Russian], Izd. AN SSSR, Moscow (1962), 252 p. (the latter is written in cooperation with V. A. Zalgaller); and his achievements concerning polyhedra are collected in the monograph “Convex Polyhedra” [in Russian], GITTL, Moscow–Leningrad (1950), 428 p.³

Many results by Alexandrov are part and parcel of the monographs by A. V. Pogorelov, I. Ya. Bakelman, Yu. G. Reshetnyak, et al.

Alexandrov elaborated a new direction in the theory of differential equations, geometric theory of fully nonlinear elliptic equations. It is reflected in his publications of which we mention a few:

(1) Additive Set-Functions in Abstract Spaces, I–IV, *Mat. Sb.*, **8** (50), issue 2, 1940; **9** (51), issue 3, 1941; **13** (53), issues 2–3, 1943.

(2) Existence and Uniqueness of a Convex Surface with a Given Integral Curvature, *Dokl. AN SSSR*, **35**, 1942.

(3) Uniqueness Theorems for Surfaces “in the Large,” parts I–III, *Vestnik Leningrad Univ. Mat.*, **19** 1956, pp. 5–14; **7** 1957, pp. 15–44; **7** 1958, pp. 14–26.

(4) Studies of the Maximum Principle, I–IV, *Izv. Vuzov Mat., Kazan'*, **5** 1958; **3**, **5** 1959; **3**, **5** 1960; **1** 1961.

(5) The Dirichlet Problem for the Equation $\det \|z_{x_i x_j}\| = \varphi(x_i, z, z_{x_i})$, *Vestnik Leningrad Univ. Mat.*, **1** 1958.

(6) A Uniqueness Condition and Estimates for a Solution to the Dirichlet Problem, *Vestnik Leningrad Univ. Mat.*, 1963, pp. 5–29.

In the works pertinent to this direction solution is given to a number of geometric problems of surface theory “in the large” (for instance, the titles of items (2) and (5) indicate the problems solved in these articles). Of not lesser import is the fact that they established theorems on elliptic operators which are one of the principal constituents of the modern solvability theory for fully nonlinear equations of elliptic type (cf. monographs by A. V. Pogorelov, I. Ya. Bakelman, N. V. Krylov and Yu. G. Reshetnyak; articles by O. A. Ladyzhenskaya, N. N. Ural'tseva, N. M. Ivochkina and many foreign mathematicians).

We briefly present the formulations of Alexandrov's theorems giving a general idea of his achievements.

(1) **Theorem 1 (of gluing).** *Let F_1 and F_2 be two convex surfaces in \mathbb{R}^3 , homeomorphic with the disk and bounded by the curves γ_1 and γ_2 of the same length. Let a one-to-one correspondence be given between the points of γ_1 and γ_2 which preserves the arclengths of these curves and such that the sum is nonnegative of the geodesic curvatures of γ_1 and γ_2 at the corresponding points of the surfaces F_1 and F_2 . Then there is a closed convex surface F composed of two parts, one isometric with F_1 and the other isometric with F_2 .*

A. V. Pogorelov in his monograph “Flexing of Convex Surfaces” [in Russian], GITTL, Moscow (1951) writes: “One of the most powerful tools for study of flexing of convex surfaces is the gluing method based on the next remarkable theorem”; Theorem 1 follows the citation (see p. 9).

(2) **Theorem 2.** *Each polyhedral metric of nonnegative curvature, given on the two-dimensional sphere, is realizable as a convex polyhedron; and each metric of nonnegative (integral) curvature, given on the two-dimensional sphere, is realizable as a closed surface in \mathbb{R}^3 .*

(3) In 1942 Alexandrov proved

Theorem 3. *Let $\Psi(m)$ be a nonnegative countably additive function of Borel sets m on the plane $(x, y) \in \mathbb{R}^2$ which satisfies the inequality $\Psi(\mathbb{R}^2) \leq 2\pi$. Then, over the whole plane \mathbb{R}^2 , there is a convex surface F with a unique projection to \mathbb{R}^2 such that for every Borel set $M \subset F$ its extrinsic curvature equals Ψ on the projection of the set M to the plane \mathbb{R}^2 .*

²A translation into English is now in preparation by Gordon and Breach.

³A translation into English is now in preparation by Springer-Verlag.

He also proved an analogous existence theorem for a convex surface with a given nonnegative curvature (which is an analog of the Gaussian curvature in the case of smooth realization).

(4) Of the profusion of the theorems proven by Alexandrov for arbitrary convex surfaces (and further for surfaces with a unilateral curvature constraint) we cite only two:

Theorem 4 (generalization of the Gauss–Bonnet theorem). *The extrinsic curvature of a convex surface equals its intrinsic integral curvature.*

Theorem 5 (on comparison of the angles of geodesic triangles). *The angles of every small geodesic triangle on a metric manifold of nonnegative curvature are not less than the corresponding angles of the triangle in the Euclidean space with sides of the same lengths.*

This theorem and its generalizations play an important role in study of various metric spaces of arbitrary dimension (cf. the “ K -spaces of Alexandrov”).

(5) **Theorem 6.** *Let $F^{(1)}$ and $F^{(2)}$ be two closed convex surfaces in \mathbb{R}^3 , and let at the points $x^{(k)} \in F^{(k)}$, $k = 1, 2$, with parallel normals \vec{n} , the principal curvatures $k_j^{(i)}$, $i, j = 1, 2$, of the surfaces, indexed in decreasing order, satisfy the equality*

$$f(k_1^{(1)}, k_2^{(1)}, \vec{n}) = f(k_1^{(2)}, k_2^{(2)}, \vec{n}),$$

where f is a given function increasing in k_1 and k_2 . Then $F^{(1)}$ and $F^{(2)}$ are translates of one another.

For the case of $F^{(k)}$ analytic, this theorem was proven by Alexandrov in 1938. In 1956 he relaxed the requirement to the second order differentiability of $F^{(k)}$, while replacing the convexity condition for analytic surfaces with the condition that they be homeomorphic with the sphere. He established various uniqueness theorems also for surfaces in \mathbb{R}^n with any $n \geq 3$ and in Riemannian spaces as well. Among them there is, for instance, a theorem claiming that a surface in \mathbb{R}^n having constant positive mean curvature and presenting the boundary of some body is a sphere. As many of the theorems by Alexandrov, this theorem is “exact”: its last hypothesis is impossible to eliminate.

(6) In 1958 Alexandrov gave an original reformulation of the Dirichlet problem for the equations of Monge–Ampère type

$$f(x, z, z_x) \det z_{xx} = h(x), \quad x \in \Omega \subset \mathbb{R}^n$$

and found their generalized solutions in the class of convex functions. This remarkable study too gained diverse applications in the works by many mathematicians. (A. V. Pogorelov, N. M. Ivochkina, N. V. Krylov, P. Lions, L. Caffarelli, L. Nirenberg, Sprock, et al.).

(7) Not lesser influence was and is still exerted by the works of Alexandrov on the maximum principle and estimation of solutions to linear and nonlinear elliptic equations.

We present one of his estimates here. For an arbitrary function $u \in W_n^2(\Omega) \cap C(\bar{\Omega})$ in an arbitrary domain $\Omega \subset \mathbb{R}^n$ it assumes the form

$$\max_{x \in \Omega} u(x) \leq \max_{x \in \partial\Omega} u(x) + c_1 \text{diam } \Omega \exp\{c_2 |b|_{n,\Omega}\} |(\mathcal{L}_u) - |_{n,\Omega}.$$

Here

$$\mathcal{L}_u = \sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n b_i u_{x_i}, \quad a_{ij}(x) \xi_i \xi_j \geq 0,$$

and

$$|b|_{n,\Omega} \equiv \left(\int_{\Omega} |b(x)|^n (\det a_{ij}(x))^{-1} dx \right)^{1/n}.$$

The constants c_1 and c_2 depend only on n . This result by Alexandrov also found in-depth applications and generalizations (N. V. Krylov, V. Safonov, N. N. Ural'tseva, A. Nazarov, et al.).

It may be asserted that Alexandrov is a pioneer in establishing direct methods for solving the problems of geometry “in the large” and of nonlinear elliptic equations of geometric provenance.

NOMINATION BY RESHETNYAK

Alexandrov is nominated to the Wolf Prize for the contribution contained in the articles that established a new direction in geometry, the theory of nonregular Riemannian spaces. The central place in Riemannian geometry is occupied by the theory of the curvature of a space. Alexandrov aimed at constructing a theory of nonregular Riemannian spaces satisfying some curvature boundedness condition at least in a certain generalized sense.

The articles by Alexandrov on the theory of generalized Riemannian space provide further development of the geometric concept of space along the lines of the tradition stemming from Lobachevskiĭ, Riemann, and E. Cartan.

Alexandrov developed the theory of two-dimensional manifolds of bounded curvature. Thus, the indicated abstract problem of constructing the theory of nonregular Riemannian spaces satisfying the curvature boundedness condition is completely solved for the case of dimension two. The challenging question as to which space of greater dimension should be considered as analogs of the two-dimensional case is partially solved by Alexandrov in his theory of spaces with curvature not greater than K .

Basic for the Alexandrov theory of two-dimensional manifolds of bounded curvature are the papers devoted to solving the famous Weyl problem. The latter consists in proving that each two-dimensional Riemannian metric of positive curvature given on the sphere is realizable as the intrinsic metric of a closed convex surface. One solution to this problem was given by H. Lewy basing on entirely different arguments. Alexandrov gave another purely geometric solution. It may undoubtedly be ranked as exemplar of beauty in mathematics. The main difficulty in the Alexandrov approach resides in settling the problem for polyhedra. Its surmounting required rather subtle gadgets from the tool-kit of modern mathematics.

The general theorem on existence of a convex surface with a given metric was derived by Alexandrov from the respective theorem for polyhedra by passage to a limit. A Riemannian metric of positive curvature on the sphere (i.e. a metric defined by the element of positive Gaussian curvature), as was proven by Alexandrov, is the limit of a polyhedral metrics satisfying the following condition: Each of these polyhedral metrics is a metric of positive curvature i.e. is realizable as the intrinsic metric of some closed convex polyhedron. For a Riemannian metric of positive curvature on the sphere some sequence thus appears of closed convex polyhedra whose metrics converge to the initial metric. Alexandrov also demonstrated that this schema applies to the general case of convex surfaces in a space of constant curvature.

Alexandrov introduced the concept of two-dimensional manifold of curvature at least K and proved that such a manifold is homeomorphic with a convex surface in a space of positive curvature. Thus the problem was solved of describing the intrinsic metric of a convex surface in a space of constant curvature. That resolves the Weyl problem in a generalized setting.

MORE DETAILED DESCRIPTION FOR ALEXANDROV'S CONTRIBUTION TO GEOMETRY BY ZALGALLER

Alexandrov made fundamental contribution to research into the problems of geometry "in the large" and origination of new methods paving ways of further studies in the field for many mathematicians. We will just list the principal directions of his research (in approximately chronological order).

(1) Alexandrov advanced the theory of mixed volumes expounded by Minkowski. He particularly established the most general inequality between mixed volumes. This stimulated the modern development of interplay between the theory of mixed volumes with the theory of complex functions (Kushnirenko, Bernstein, Tessier, Khovanskiĭ, and Gromov).

(2) Alexandrov developed the theory of completely additive set functions in abstract metric spaces and the geometric theory of weak convergence of such functions. This cleared the way for introducing integral (rather than pointwise) functional characteristics in geometry and using weak convergence in the theory of conventional and signed measures.

(3) Alexandrov proved the theorem that each "development," a complex of plane polygons with identified pairs of edges of the same length, may be uniquely realized as a convex polyhedron in \mathbb{R}^3 , provided that the development is homeomorphic with the sphere as a whole and the sum of plane angles at each

vertex is at most 2π . (It is not excluded that the edges of the development are not necessarily edges of the polyhedron, appearing simply as “drawn” on it.)

The proof of this remarkable theorem is based on an especially invented method allowing one to demonstrate that a mapping of a manifold into another manifold of the same dimension is a mapping onto the whole manifold. The method (which is a far-reaching generalization of the method of continuation by a parameter) enabled Alexandrov to prove an impressive succession of general theorems on the condition that specify existence or uniqueness of a convex polyhedron with some prescribed data.

The results of this cycle of works ranked the name of Alexandrov in the same row as the names of Euclid and Cauchy.

(4) On using approximation by polyhedra, Alexandrov, solved (in a strengthened form without smoothness requirements) the Weyl problem on realizability as a closed convex surface of each metric of non-negative curvature given on the sphere.

(5) From an analytical viewpoint, in these studies Alexandrov developed the theory of generalized solutions for geometry, staying here several decades ahead of specialists in the fields of analysis and differential equations.

He paid less attention to the question of how smooth are such solutions. However, even here he was the first who proved that every convex surface has second differential almost everywhere, and if a convex surface has bounded specific curvature then the surface is C^1 -smooth.

(6) On using synthetic methods, Alexandrov studied first the intrinsic geometry of an arbitrary convex surface, and next, that of an arbitrary manifold of bounded curvature. The class of the latter in view of its compactness served as the space in which many extremal problems are resolved. This class is a kind of closure for two-dimensional Riemannian manifolds. (The two-dimensional manifolds of bounded curvature, introduced by Alexandrov, were shown by Yu. G. Reshetnyak to possess some nonsmooth Riemannian metric.)

(7) Studying the intrinsic geometry of convex surfaces, Alexandrov proved the “gluing theorem.” The latter, together with Alexandrov’s theorem of realization of convex metrics, provided a base for the modern state of the flexing theory of convex surfaces with boundary in the class of convex manifolds.

(8) Considering multidimensional metric spaces with every two points joined by a shortest arc, Alexandrov introduced a general concept of the angle between shortest arcs and, by comparing the angles of an infinitesimal triangle with those of the triangle having the sides of the same lengths on the K -plane (the two-dimensional surface of constant Gaussian curvature K), defined spaces with curvature $\leq K$ or $\geq K$. These spaces are named “Alexandrov spaces.” Observe that n -dimensional Riemannian manifolds whose all sectional curvatures K_σ satisfy the inequalities $K_\sigma \leq K$ or $K_\sigma \geq K$ are particular instances of the Alexandrov spaces.

It is exactly the angle comparison theorem by Alexandrov and its “nonlocal” generalization given by V. A. Toponogov that started the rapid development of the modern Riemannian geometry in the large (Toponogov, Klingenberg, Berge, Thurston, et al.).

(9) Similarly as “metrizable” spaces are distinguished among topological spaces, Alexandrov posed the question of selecting among metrizable n -dimensional manifolds those “Riemannizable” whose metric may be given by a quadratic linear element. Together with his students V. N. Berestovskii and I. G. Nikolaev, he proved that every manifold presenting an Alexandrov space of curvature $\geq K_1$ and simultaneously $\leq K_2$ is Riemannizable but with a metric of “lesser” smoothness.

A particular interest in Alexandrov spaces was raised by the results of M. Gromov who showed that the passage to the limit from the class of Riemannian metrics with uniformly (upper or lower) bounded sectional curvatures leads precisely to the class of Alexandrov spaces. The geometry of these spaces undergoes intensive study nowadays (cf., for instance, the article by M. Gromov, Yu. D. Burago and G. Perel'man in the “Russian Mathematical Surveys,” 1993).

The research into the intrinsic geometry of metrizable manifolds ranks the name of Alexandrov in the same row as that of Gauss and Riemann.

This part of the survey lacks any description of the achievements by Alexandrov in the field of differential equations of elliptic type.

CLOSING REMARKS

As I mentioned above, Professor Alexandrov was never awarded with the Wolf Prize nor the Lenin Prize which he would be even more pleased to receive. Although the list of his degrees, decorations, signs of honor, and other trophies is immense, he had deserved much more during his life for his outstanding efforts and monumental contribution. Moreover, I am convinced that the future generations of geometers will indulge in invoking his illuminative ideas, wise definitions, sharp inequalities, and strong theorems which compose his eternal memory.