HOMOLOGY PROPERTIES OF Y-SYSTEMS

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Suppose that a Y-system $T^t(T^k)$ acts on a manifold M^n. We present a criterion of zero homology for Hölder functions with respect to this dynamical system, as well as some consequences of this criterion and a generalization for functions taking their values in a Lie group.

Let M^m be a smooth closed Riemannian manifold of class C^2 with a metric ρ, let $T^t\,(T^k)$ be a smooth flow (cascade) [1], and f a real function on M^m that satisfies a Hölder condition; f is said to be homologous to zero in the class of Hölder functions if there exists a Hölder function g, such that

$$f(x) = \frac{d}{dt} g(T^t x)_{t=0} \qquad (f(x) = g(Tx) - g(x)).$$

The main result of this note is the following criterion of zero homology of a function.

THEOREM 1. If T^t is a Y-flow (T^k a Y-cascade) [1] with everywhere-dense trajectories, then for f to be homologous to zero in the class of Hölder functions, it is necessary and sufficient that the following condition hold for any periodic trajectory $\{T^t x\}_{t=0}^{t=\tau}$ ($\{T^n x\}_{n=1}^{n=n_0}$):

$$\int_0^\tau f(T^t x)\,dt = 0 \qquad \left(\sum_1^{n=0} f(T^n x) = 0\right), \tag{1}$$

and if the Hölder modulus of continuity of the function f is $\omega(\rho) = C\rho^\delta$, then the modulus of continuity of the function g will not exceed $C_\delta C \rho^\delta$, where C_δ is a constant that depends on δ and on the dynamical system.

Proof. The necessity is evident. Let us prove the sufficiency. For simplicity we shall consider the case of a cascade. The manifold M^m is assumed to be endowed with a Lyapunov metric which is matched with the Y-condition for our cascade [2]. Let us prove the principal lemma.

LEMMA. There exists a positive K such that for any ε there exists an N_ε, and if $n > N_\varepsilon$, then it follows from $\rho(T^n x, x) < \varepsilon$ that there exists an x_0: $T^n x_0 = x_0$ and a $\rho(T^l x_0, T^l x) < K_\varepsilon$ for $1 \le l < n$.

Proof. Let $\mathfrak{S}^k$ and $\mathfrak{S}^l$ be invariant contracting and expanding foliations, respectively. We shall use the following assertion, a consequence of the continuous dependence of the fibers (foliations) on the initial points: there exists a positive a such that if Π_0 and Π_1 are smooth areas that lie in the fibers $\mathfrak{S}^k$ and such that any point $\omega_0 \in \Pi_0$ can be connected by a path of length $< a$ that lies in a fiber $\mathfrak{S}^l$ with a point $\omega_1 \in \Pi_1$, then the mapping $U: \Pi_0 \to \Pi_0$, $u(\omega_0) = \omega_1$ will be continuous, and in the case of a small continuous deformation of these areas this mapping will vary continuously ([1], p. 26). Hence, it follows from the compactness of M^m that all the U constructed in this way have moduli of continuity that do not exceed a common $\delta(\varepsilon)$ ($\delta(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$), where an induced metric is taken in the fibers. Moreover, there exist positive γ and C such that for any two points A and B of the manifold M^m with $\rho(A, B) < \gamma$ it is possible to find a point S that lies in one contracting fiber containing A and in one expanding fiber containing B, with the distance from S to A and from S to B in the fibers being smaller than $C\rho(A, B)$.

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Now let $\rho(x, T^n x) < \varepsilon$. For $x_1 T^n x$ we shall consider the point $S_{x_1 T^n x}$. About the point x in a contracting fiber we shall describe a sphere D of radius $(C+1)\varepsilon$. If ε is smaller than some fixed quantity, whereas n is sufficiently large, then (firstly) according to the Y-property the diameter of the image of $T^n D$ will be sufficiently small for defining a mapping U that carries $T^n D$ into a contracting fiber that passes through the point x, and (secondly) for a given ε it is possible to take n so large that $U(T^n D)$ will lie in a sphere of radius ε with its center at the point S in a contracting fiber, or $U(T^n x) = S$. From the presence of a "universal" modulus of continuity for U follows the presence of a universal N_ε that can be taken as a lower limit for such n. But a sphere of radius ε with center at the point S is contained in the initial sphere of radius $(C+1)\varepsilon$ with center at the point x. Hence, according to Browder's theorem there exists in a circle of radius $(C+1)\varepsilon$ with center at the point x a fixed point of the mapping $U \circ T^n$. This point has the property that its image under a mapping T^n lies in an expanding fiber at a distance from it that is smaller than $C'\varepsilon$ along the expanding fiber; here, C' is a constant. In this fiber, T^{-n} is a contraction mapping. Therefore, there exists a fixed point of the mapping T^n at a distance from it that is smaller than $C''\varepsilon$. It is evident that this will be the periodic point sought. This completes the proof of the lemma.

Now let $\{T^k x\}_1^\infty$ be an everywhere-dense trajectory that exists by virtue of the assumption of the theorem. On it let us assign a function g: $g(T^k x) = \sum_0^{k-1} f(T^l x)$. We have to prove that there exists a positive L such that

$$| g(T^{l'}x) - g(T^l x) | < L\rho(T^{l'}x, T^l x)^\delta, \tag{2}$$

where $0 < \delta \leq 1$ (we shall prove that δ can be taken as the Hölder index of the function f). By virtue of continuity it is then possible to continue g to all M^m in a Hölder manner, and the continuation will be the function sought. For the proof we shall assume at first that $|l - l'| > N_{\rho(T^{l'}x, T^l x)}$ (in the notation of the lemma). By virtue of the lemma there exists in this case an x' such that $T^{l'-l}x' = x'$ (let $l' > l$) and $\rho(T^{k-1}x, T^{k-l-1}x') < K\rho(T^l x, T^{l'}x)$ for $l + 1 \leq K \leq l'$. Let us consider the difference $g(T^{l'}x) - g(T^l x) = \sum_{l+1}^{l'} f(T^k x)$. By virtue of the assumption of the theorem, it is equal to

$$\sum_{l+1}^{l'} f(T^{k-1}x) - \sum_0^{l'-l-1} f(T^m x^1) = \sum_{l+1}^{l'} [f(T^{k-1}x) - f(T^{k-l-1}x^1)]. \tag{3}$$

To each pair $T^{k-1}x$ and $T^{k-l-1}x^1$ we shall assign a point S_k, as in the proof of the lemma; if $\rho(T^l x, T^{l'}x)$ is sufficiently small, it is possible to assume that $S_{k+1} = TS_k$ for any k, and each difference (3) can be represented in the form $[f(T^{k-1}x) - f(S_k)] + [f(S_k) - f(T^{k-l-1}x^{-1})] \leqslant L'\rho(T^{k-1}x, S_k)\,\delta + L'\rho(S_k, T^{k-l-1}x^1)^\delta$, where L' is Hölder's constant of the function f. But

$$\sum_{l+1}^{l'} L'\rho(T^{k-1}x, S_k)^\delta < \sum_{l+1}^{\infty} L'\rho(T^{k-1}x, S_k)^\delta \leqslant L''\rho(T^l x_1 S_{l+1})^\delta \leqslant L'''\rho(T^l x, T^{l'}x)^\delta.$$

The convergence and the bound of the series follow from the fact that $T^{k-1}x$ and S_k lie in the same contracting fiber, and hence, the general term of the series varies exponentially. In the same way it is possible to estimate the sum of the second terms of (3). Thus, we have proved for the case under consideration the bound (2). For arbitrary l and l' the proof reduces to the following: since the trajectory of the point x is everywhere dense, there exist (for arbitrarily large N) quantities $n > N$ with $T^n x$ that are as close as desired to $T^l x$, and (2) can be applied to $T^l x$ and $T^n x$, as well as $T^{l'}x$ and $T^n x$. This complete the proof of the theorem. The proof for flows is basically the same, apart from the formulation of the lemma. For flows, the lemma will be as follows.

<u>LEMMA</u>. There exist K_1 and K_2 such that for any ε there exists an N_ε, and if $t > N_\varepsilon$, then from $\rho(T^t x, x) < \varepsilon$ it follows that there exist x_0 and t_0 with $|t_0 - t| < K_1\varepsilon$; $\rho(T^s x, T^s x_0) < K_2\varepsilon$ for $0 \leq s \leq \min(t_0, t)$ and $T^{t_0}x_0 = x_0$.

The proof is entirely analogous. The origin of K_1 is as follows. If $T^t x$ is close to x, then there exists a t_0 close to t, such that $T^{t_0}x$ can be connected with x by a two-section broken line that has one contracting and one expanding section.

Remark 1. Let us consider a continuous function f on a manifold M^m that satisfies the condition of convergence of the integral $\int_0 \frac{\omega(\delta)}{\delta} d\delta$, where $\omega(\delta)$ is the modulus of continuity of f.

In this case the vanishing of sums over periodic trajectories in the case of a cascade, and of integrals over periodic trajectories in the case of a flow, implies that f is homologous to zero in $C(M^m)$, which follows from the proof of Theorem 1.

It is evident that the function g is also obtained in this case by continuation from an everywhere-dense trajectory.

Remark 2. The zero homology of a function that satisfies the condition of Remark 1 in the class of essentially bounded measurable functions implies its zero homology in $C(M^m)$. Indeed, suppose that for a function f that satisfies the condition of Remark 1 there exists an essentially bounded measurable function such that $f(x) = g(Tx)-g(x)$ for almost all x (for simplicity we are considering the case of a cascade). By assuming that for a periodic trajectory $\{T^k x_0\}_1^n$, we have $\sum_1^n f(T^k x_0) = c \neq 0$, we obtain $\left|\sum_1^{mn} f(T^k x_0)\right| = |mc| \leqslant v\,\mathrm{rai\ sup}\left|\sum_1^{mn} f(T^k x)\right|$, owing to the continuity of f, but $\sum_1^{mn} f(T^k x) = g(T^{mn+1}x) - g(x)$, i.e., by letting m tend to infinity, we can see that the condition of essential boundedness of g is violated. Thus, the sums over periodic trajectories are equal to zero, and according to Remark 1 the function f is homologous to zero in C.

Remark 3. If M is a torus and T its automorphism or endomorphism, we find that for a function f that satisfies Hölder's condition and the condition of absolute convergence of the Fourier series, (zero homology in $L'(M, \mu)$, where μ is Lebesgue's measure) will be equivalent to zero homology in C. Indeed, if f is homologous to zero in L' and there exists a $g \in L'$ such that for almost all x we have $f(x) = g(Tx)-g(x)$, then, by considering the Fourier series of the function g with respect to the characters of a torus, we can see that in an expansion of the function f, the sum of the Fourier coefficients over any complete trajectory of an endomorphism (conjugate to T), of a group of characters, will be equal to zero.

Let $\{T^k x_0\}_1^m$ be a periodic trajectory of T in a torus. Let us prove that $\sum_1^m f(T^k x_0) = 0$. In fact, to any complete trajectory of T^* in the group of characters of a torus (T^* being the conjugate endomorphism) there corresponds $S = \{T^{*p}\gamma\}_0^\infty$ (or $S = \{T^{*p}\gamma\}_{-\infty}^{+\infty}$); by taking the function $\varphi_S = \sum_{T^{*p}\gamma \in S} a_p^S T^{*p}\gamma$, where the a_p^S are the Fourier coefficients of f for $T^{*p}\gamma$, we obtain

$$\sum_1^m \varphi_S(T^k x_0) = \sum_1^m \sum_{T^{*p}\gamma \in S} a_p^S T^{*p}\gamma(T^k x_0)$$

$$= \sum_{p:T^{*p}\gamma \in S} a_p^S \sum_{p+1}^{p+m} \gamma(T^k x_0) = \sum_{p:T^{*p}\gamma \in S} a_p^S \sum_1^m \gamma(T^k x_0) = 0$$

(since $T^m x_0 = x_0$), but since $f = \sum_S \varphi_S$, we have also for f the relation $f: \sum_1^m f(T^k x_0) = 0$. It hence follows from Remark 1 that f is homologous to zero in C.

THEOREM 2. Zero homology of a continuously differentiable function f in the class of essentially bounded measurable functions is equivalent to zero homology in the class of continuously differentiable functions.

Proof. According to Remark 2 the function f is homologous to zero in C, and the corresponding g can be obtained by continuation from an everywhere-dense trajectory T^n of a cascade (for convenience we confine ourselves also here to the case of a cascade). Now let us consider a point A, as well as a "contracting" vector X of unit length, and a smooth curve $a(t)$; $a(0) = A$; $0 \leq t \leq t_0$; A(t) lies in a contracting fiber, and the vector tangent to $a(t)$ at the point A for t = 0 is X. Let $\{T^k x\}_0^\infty$ be an everywhere-dense trajectory from which we continue the function g. By taking $t' \in [0, t_0]$, we obtain pairs (n, n') of natural numbers with an arbitrarily large difference $n'-n$ such that $T^n x$ is as close as desired to A, and $T^{n'} x$ is as close as desired to $a(t')$. If t' is sufficiently small, there will exist for a sufficiently large n'−n a periodic point A' of period n'−n that approximates the point $T^{n'}x$ according to the lemma; from the proof of the lemma we can see that by taking n'−n sufficiently large, $T^n x$ sufficiently close to $a(0) = A$, and $T^{n'}x$ sufficiently close to $a(t')$, it is possible to achieve that our periodic point A' is close to $a(t')$. Then the difference $g(a(t'))-g(A)$ will be close to

$$g(T^{n'}x) - g(T^n x) = \sum_n^{n'-1} f(T^k x) = \sum_n^{n'-1} [f(T^k x) - f(T^{k-n}A')].$$

By letting n'−n tend to $+\infty$ by varying n and n' in such a way that $T^n x$ tends to A and $T^{n'}x$ to $a(t')$, we can achieve that our difference will tend, on the one hand, to $g(a(t'))-g(A)$, and on the other hand to

$\sum_0^\infty [f(T^k A) - f(T^k(a(t')))]$ (this series converges in view of the fact that A and $a(t)$ lie in the same contracting fiber, and f is differentiable). Suppose that the vector tangent to $a(t)$ at the point A is $\xi(t)$; hence,

$f(T^k A) - f(T^k(a(t'))) = -\int_0^{t'} \widetilde{T}^k(\xi(t)), df T^k(a(t))$, where $\widetilde{T}^k$ is the differential of a diffeomorphism T^k. Since

$\xi(t)$ is a "contracting" vector, we find at once that g is differentiable in the direction $\xi(0) = X$, and that the

derivative is equal to $-\sum_0^\infty (\widetilde{T}^k(X), df(T^k A))$. In the same way we obtain differentiability along any "expanding" inequality, by evidently considering T^{-1}. From the transversality of contracting and expanding foliations, and from the continuous dependence of fibers on the initial data, it is now easy to obtain the differentiability of the function g (taking into account the uniform convergence of the series for the derivative).

Remark. In the general case it is apparently not evident that the function g can be differentiated as many times as f, even if we assume that f is only twice differentiable. Yet nevertheless, for semisimple ergodic endomorphisms of the torus (i.e., for endomorphisms whose matrices have one-dimensional Jordan cells) we have the following result. If $f \in C^{r+\delta}$, where $\delta \leq 1$ and all the r-th derivatives have absolutely convergent Fourier series, then the zero homology of f in C implies its zero homology in $C^{r+\delta}$. Let us outline the proof of the existence of mixed derivatives with respect to characteristic directions of an endomorphisms (this is sufficient). If the characteristic directions selected by us are such that the products of the corresponding eigenvalues are different from unity, it is possible to prove the existence of a mixed derivative in the same way as in Theorem 2 (the corresponding series converges, of course, to a Hölder function with index δ). If the product of the eigenvalues is equal to unity, i.e., the series diverges, we shall denote by D the corresponding differentiation operator. If $g(Tx)-g(x) = f(x)$ and if there exists a continuous function Dg, we obtain (in view of the fact that the product of the eigenvalues is equal to unity) the formula

$$Dg(Tx) - Dg(x) = Df(x).$$

This means that we must prove first of all that Df is homologous to zero in C. Let us consider the Fourier series and Df. From the fact that the product of eigenvalues is equal to unity, it easily follows that these series differ along the trajectories of an endomorphism (conjugate to T) of the group of characters of the torus by factors that are constant on the trajectories; therefore, the sums of the coefficients along the trajectories are equal to zero in f and Df simultaneously. By acting in the same way as in Remark 3 to the previous theorem, we can see that the function Df is homologous to zero in C, and hence, in the space of Hölder functions. Among the functions realizing this homology, we shall select the function that has in the Fourier expansion a zero coefficient in a trivial character. This function will be precisely Dg. Indeed, by formally applying the operator D to the Fourier series of the function g, the obtained series will likewise realize (in the space of formal Fourier series) a zero homology of f. Therefore, the difference between them and the Fourier series of the function obtained by us will be constant on the trajectories of the endomorphism of the group of characters, conjugate to T. But this constant will be zero on each trajectory, since the Fourier coefficients of the function g and of the formal series Dg differ by factors that are constant on the trajectories, and hence, they tend simultaneously to zero along the trajectories in the same way as in the Fourier series of our "candidate" in Dg, and hence, also in the same way as in the difference under consideration. Thus, Dg exists by virtue of ordinary theorems of differentiability of uniformly convergent functional series. Its Hölder property with index δ follows from Theorem 1.

It is of interest to study homologies of Y-systems with coefficients in a Lie group. Suppose that a group G acts on a Riemannian manifold M^m, and that Γ is a Lie group. Let $\Gamma(M)$ denote a group of Hölder mappings of M into Γ (the Hölder property does not depend on the metric, in view of the compactness of M). A cycle is a function $f(x, g)$ that maps $M \times G$ into Γ and such that

$$f(x, g_1) f(xg_1, g_2) = f(x, g_1 g_2).$$

Two cycles f_1 and f_2 are said to be homologous if

$$f_1(x, g) = \varphi^{-1}(x) f_2(xg) \varphi(xg)$$

for a Hölder function φ: $M \to \Gamma$. In the case G = Z, a cycle will be defined by the function $\bar{f}(x) = f(x, 1)$, and the homology condition will be

$$\bar{f}_1(x) = \varphi^{-1}(x) \bar{f}_2(x) \varphi(Tx),$$

where T is the action of a generating element of the group G. If G is an additive group of real numbers and if we require the differentiability of cycles and of the function $\varphi(xg)$ with respect to g, we shall define a cycle by the function $\bar{f}$: $M \to \mathfrak{G}$ ($\mathfrak{G}$ is a Lie algebra of the group Γ):

$$\bar{f}(x) = \lim_{t \to 0} t^{-1} \exp^{-1} [f(x, 0)^{-1} f(x, t)],$$

where $\exp^{-1}$ is defined for small t, and the condition of zero homology of a cycle will be expressed in terms of $\bar{f}$ as follows:

$$\bar{f}(x) = \lim_{t \to 0} t^{-1} \exp^{-1} (\varphi^{-1}(x) \varphi(T^t x)),$$

where T^t is a flow defined by the action of the group R and φ is a function

$$\varphi: M \to \Gamma.$$

Thus, we can justify the definitions preceding Theorem 1, and we can refer to functions that are homologous to zero. Let us formulate without proof a theorem that generalizes Theorem 1.

THEOREM 3. If the group G is either Z or R and its action on a Riemannian manifold M^m specifies a Y-cascade (or Y-flow) with an everywhere-dense trajectory, there will exist in any Lie group Γ a neighborhood of unity U (in the Lie algebra of the group Γ there exists a neighborhood of zero U) such that the Hölder function

$$\bar{f}: M \to U$$

defines a null element $H'(G, \Gamma(M))$ if and only if for the corresponding cycle $f(x,g)$ $xg = x$ implies $f(x,g) = e_\Gamma$.

Remark 1. If the group Γ has a two-sided invariant metric or is finite-dimensional, then we can take U in the form of the entire group Γ (the entire Lie algebra of the group Γ). For the action of Z it is sufficient that Γ should have a two-sided invariant metric, which is not necessarily a Lie group.

Remark 2. It is evident that the condition of Theorem 3 is equivalent to the fact that e_Γ is equal to products over periodic trajectories as in Theorem 1. In the case of a two-sided invariant metric in the group, the proof of Theorem 3 will be a simple repetition of the proof of Theorem 1.

In the proof of Lemma 1 on approximation of a trajectory by periodic trajectories, we used the remarks made by G. A. Margulis.

Theorem 3 constitutes the answer to the question posed by A. M. Vershik in connection with Theorem 1.

LITERATURE CITED

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