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A FUNCTIONAL LIMIT THEOREM FOR THE POSITION OF A PARTICLE IN A LORENTZ TYPE MODEL

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Abstract. Consider a particle moving through a random medium, which consists of spherical obstacles, randomly distributed in $\mathbb{R}^3$. The particle is accelerated by a constant external field; when colliding with an obstacle, the particle inelastically reflects. We study the asymptotics of $X(t)$, which denotes the position of the particle at time t, as $t \to \infty$. The result is a functional limit theorem for $X(t)$.

Key word and phrases: Lorentz model, motion in random medium, functional central limit theorem for Markov chains, limit theorems.

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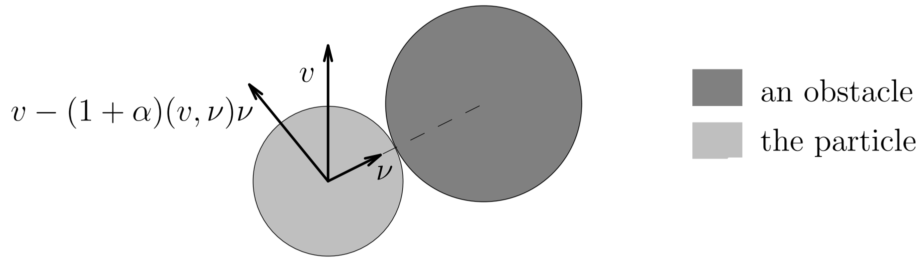
1. The Lorentz model and the problem

1.1. **The motivation.** Consider a spherical particle moving in a random medium. The medium consists of immobile spherical obstacles of equal radii, randomly distributed in $\mathbb{R}^3$. The particle is accelerated by an external field providing constant acceleration a. At a collision with an obstacle, the particle's speed v changes to

$$v - (1+\alpha)(v,\nu)\nu,$$

where $0 \le \alpha \le 1$ is the restitution coefficient and ν is the inner unit normal to the obstacle at the point of collision.



This mapping changes only the normal component of v, i.e., $(v,\nu)\nu$, which is multiplied by $-\alpha$.

This model is often named after Hendrik Lorentz who introduced it (see [8]) in order to describe conductivity in metals. Lorentz studied the case of elastic collisions, with $\alpha = 1$; the generalization for non-elastic collisions could be found, e.g., in [15].

In physics the Lorentz model is used to describe the motion of a particle in a medium if the particle's mass is negligible with respect to masses of the medium's particles (obstacles). Indeed, in this case we can assume that the obstacles have infinite masses and thus they remain immobile at collisions. For example, the model in question describes well the motion of electrons in helium, see [16].

In case of elastic collisions, the Lorentz model is a billiard type model; recall that *a billiard* is the following model of motion of a particle in an arbitrary region with smooth boundary: at a collision with the boundary, the particle elastically reflects. Indeed, we can consider the region whose boundary consists of the obstacles' boundaries. The case when the obstacles are located periodically, is especially interesting (see [4], [12]); such billiards are called *Lorentz periodic gases*. The main tool for studying billiards is ergodic theory; the basic results of this theory and their applications for the Lorentz model could be found in [5]. For a detailed review of problems and methods of the billiard theory, see [5] and [13].

Another interesting interpretation of the Lorentz model could be found in [15] (also, see the references therein): a particle *percolates* through an immobile medium under the constant gravity. Considering a big number of such percolating particles and neglecting interactions between them, we get a model of mixing of two dry substances, for example, powders.

There are many physical papers on the Lorentz model. Usually their main goal is to study the Boltzmann equation (that is the equation for density $p(v,t)$ of probability that at time t the particle's speed is v; this equation could be derived from the law of conservation of matter); see [1], [3], and [9]. For instance, the purpose of [9] is to prove the existence of the stationary, i.e., independent of t, solution of the Boltzmann equation in case $\alpha < 1$. It is typical that in [9] the convergence of $p(v,t)$ to the stationary solution, as $t \to \infty$, is not discussed.

1.2. **The model.** Since the Lorentz model is extremely complicated to analyze, we replace it with a simpler one. In doing so, we follow multiple papers on the Lorentz model, for example, [1], [3], [9], and [15]. For the case of elastic collisions, our simplified model coincide with one introduced in [11]; in this paper the authors only consider $\alpha = 1$ and formulate their model in a rather different way. We also note that the present simplified model was implicitly used in [9]. However, in [9] and [11] the authors do not discuss how the simpler model is derived from the original one. In Section 2 we give such explanation.

Let us formulate the simplified model. Denote by V_n the speed of the particle just before the nth collision and denote by τ_n the random time between the nth and the $(n+1)$th collisions. Let $\{\sigma_n\}_{n\geq 1} \subset S^2 \subset \mathbb{R}^3$ be uniformly distributed unit vectors; let $\{\eta_n\}_{n\geq 0}$ be exponential random variables with mean λ; and let $\{\sigma_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 0}$ be independent. Here $\lambda > 0$ is a parameter signifying the mean free path of the particle. We state that

$$V_{n+1} = V_n - \frac{1+\alpha}{2}\big(V_n + |V_n|\sigma_n\big) + a\tau_n; \tag{1}$$

$$\tau_n = F\Big(V_n - \frac{1+\alpha}{2}\big(V_n + |V_n|\sigma_n\big), \eta_n\Big), \tag{2}$$

where $F : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ is a deterministic function defined as the solution of the equation

$$\int_0^{F(v,t)} |v + as| ds = t.$$

In addition, we suppose that at time zero the particle's speed is nonrandom and equals some $v_0 \in \mathbb{R}^3$. Therefore $V_1 = v_0 + a\tau_0$, where $\tau_0 = F(v_0, \eta_0)$ is the random moment of the first collision. We stress that V_n form a Markov chain, and the motion of the particle is completely defined by this chain.

Let us briefly compare the new model and the original one. In the new model, at the nth collision the speed V_n changes to $V_n - \frac{1+\alpha}{2}(V_n + |V_n|\sigma_n)$. This corresponds to the collision with an obstacle, whose inner normal ν_n directed along the bisectrix of the angle between σ_n and V_n (in fact, $(V_n, \nu_n)\nu_n = \frac{1}{2}(V_n + |V_n|\sigma_n)$). Further, since for every v the function $F(v, \cdot)$ is monotone, we have $F\left(v, \int_0^t |v + as| ds\right) = t$. Substituting $v := V_n - \frac{1+\alpha}{2}(V_n + |V_n|\sigma_n)$ and $t := \tau_n$, and comparing the resulting equality with (2), we get

$$\eta_n = \int_0^{\tau_n} \left| V_n - \frac{1+\alpha}{2}(V_n + |V_n|\sigma_n) + as \right| ds.$$

Thus η_n is the length of the path passed by the particle between the nth and the $(n+1)$th collisions. As the mean free path is λ, it is quite natural that η_n are exponential r. vs. with mean λ. If we know the path's length η_n and the initial speed $V_n - \frac{1+\alpha}{2}(V_n + |V_n|\sigma_n)$, we can find the time τ_n; this argument explains (2).

However, there are quite significant disparities between the models. In the new model, if the particle collides with an obstacle at some point of space, then the particle does not necessarily collide again while coming back to the same point. In other words, after a collision the obstacle instantly "disappears". This happens because of the postulated independence of all σ_n, η_n. Thus the medium "changes" in a rather specific way. See more about the disparities in Section 2.

1.3. **The problem and the results.** We study the asymptotics of $X(t)$, which denotes the position of the particle at time t, as $t \to \infty$. The case $\alpha = 1$ was investigated in [11], where the authors proved that after a proper normalization the trajectories of $X(t)$ weakly converge to a certain diffusion process.

Let us mention about a very similar model of motion in $\mathbb{R}^1$: at a collision, the particle's speed v always changes to $-\alpha v$. If there is no external field and $\alpha = 1$, then the times between collisions are i.i.d. exponential r.vs., thus $X(t)$ is a well known telegraph process. This simplest model of motion is studied in detail in [7].

In the current paper, we consider the motion in $\mathbb{R}^3$, with an external field, and assume that $\alpha \in (0, 1)$. The purpose of this paper is to prove a functional limit theorem for $X(t)$, and thus to sharpen the results of the previous work [14]. We also note that the model in question could be easily generalized to obtain the model of motion in $\mathbb{R}^d$.

Without loss of generality we assume that $X(0) = 0$. Consider an orthonormal basis of $\mathbb{R}^3$ such that for the acceleration a it is true that $a = (0, 0, |a|)^\top$.

Our main result is the following

Theorem 1. *Suppose $0 < \alpha < 1$ and $|a| \neq 0$; then there exist constants $c_1 > 0$ and $c_2, c_3 \geq 0$ such that for any initial speed $v_0 \in \mathbb{R}^3$, in the space $\mathcal{C}\big([0,1], \mathbb{R}^3\big)$*

$$Y_t(s) := \frac{X(st) - c_1 ast}{\sqrt{t}} \xrightarrow{d} Y(\cdot) := \begin{pmatrix} c_2 W_1(\cdot) \\ c_2 W_2(\cdot) \\ c_3 W_3(\cdot) \end{pmatrix}, \qquad t \to \infty,$$

where W_1, W_2, and W_3 are independent Wiener processes.

Remark. *The constants c_1, c_2, and c_3 depend on the model's parameters a, α, and λ. The author failed to find these constants in an explicit form.*

Remark. *We can easily extend the model defined by the relations (1) and (2) to get the model of motion in random medium in $\mathbb{R}^d$. Indeed, let σ_n be uniformly distributed on $S^{d-1} \subset \mathbb{R}^d$, let $a \in \mathbb{R}^d$, and let $F : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ be defined as above. For this model in $\mathbb{R}^d$, Theorem 1 also holds; the proof is almost the same as for $\mathbb{R}^3$. For the limit process, it is true that $Y = (c_2 W_1, \dots, c_2 W_{d-1}, c_3 W_d)^\top$, where W_i are independent Wiener processes.*

Moreover, for the motion in $\mathbb{R}^1$, Theorem 1 is valid for the similar model, in which at a collision the particle's speed v always changes to $-\alpha v$.

A nonrigorous explanation of our result could be found in [15].

Proving this theorem, we reduce the problem to some statements about a certain Markov chain. The most difficult one is that for this chain the functional central limit theorem (FCLT) holds. The difficulties arise because the chain has uncountable and noncompact state space; moreover, the chain does not satisfy the Doeblin condition, thus the classical results (for instance, from [6]) are not applicable. We solve the problem using quite recent results (see [10] and references therein), based on stochastic analogues of Lyapunov's functions.

Therefore, our methods differ significantly from those of [11]. Nevertheless, some similarities could be found on a deep level. Indeed, the proof from [11] is based on martingale theory, while the presented one follows from the FCLT for Markov chains. Recall that the FCLT is proved by martingale theory arguments.

2. The deduction of the simplified Lorentz model

In this section we consider the original Lorentz model, described in Subsection 1.1. For this model, we find the distribution of the time before the fist collision τ_0 and the distribution of the random normal ν_1, which describes the first collision. Then we use these results to derive the simplified model, defined by (1) and (2).

Since the medium is assumed to be isotropic, it is natural to suppose that the obstacles' centers form a Poisson point process, with the control measure proportional to Lebesgue measure λ_3. The only shortcoming of this assumption is that obstacles can intersect. We denote this Poisson process by Π and write its control measure in the form $(\pi\lambda r^2)^{-1}\lambda_3$, where r is the sum of radii of the particle and an obstacle; $\lambda > 0$ is a parameter. We will see that λ signifies the mean free path of the particle.

In addition, we assume that for the initial speed v_0 the following condition holds:

$$|v_0^\perp|^2 > r|a|, \tag{3}$$

where $v_0^\perp$ is the projection of v_0 on the orthogonal complement of a.

2.1. **The distribution of τ_0.** We start with the following notations: for a $v \in \mathbb{R}^3$, define the unit hemisphere

$$\mathcal{S}_v := \left\{u \in \mathbb{R}^3 : |u| = 1, (u, v) \geq 0\right\};$$

for any $0 \leq t_1 < t_2$, define the set

$$\mathcal{A}_{t_1,t_2} := \left\{v_0 s + \frac{as^2}{2} + r\mathcal{S}_{v_0+as},\ s \in [t_1, t_2)\right\}.$$

Let us fix a $t > 0$. The inequality $\tau_0 > t$ is equivalent to the absence of obstacles' centers in the set $\mathcal{A}_{0,t}$, whence

$$\mathbb{P}\{\tau_0 > t\} = \mathbb{P}\{\Pi(\mathcal{A}_{0,t}) = 0\} = \exp\Big\{-(\pi\lambda r^2)^{-1}\lambda_3(\mathcal{A}_{0,t})\Big\}.$$

We claim that for volume of the set $\mathcal{A}_{0,t}$, which has the form of a curved "cylinder",

$$\lambda_3(\mathcal{A}_{0,t}) = \pi r^2 \int_0^t |v_0 + as|ds \tag{4}$$

(the naive explanation is the following: the factor of the integral is area of the "cylinder's" cross-section, and the integral is the "cylinder's" length). Then, for the distribution function of τ_0,

$$\mathbb{P}\{\tau_0 > t\} = e^{-\lambda^{-1}\int_0^t |v_0+as|ds}. \tag{5}$$

Also, $\mathbb{P}\{\tau_0 \le dt\} = \lambda^{-1}|v_0|dt + o(dt)$, as $dt \to 0$, thus we see that the parameter λ signifies the mean free path of the particle.

Let us prove (4). Without loss of generality, assume that $v_0^\perp = (0, |v_0^\perp|, 0)^\top$; since $a = (0, 0, |a|)^\top$, the first coordinate of the trajectory of the curve $s \mapsto v_0 s + as^2/2$ is zero. Denote by $L(t) := \int_0^t |v_0 + as|ds$ the length of this curve, and let $s \mapsto (0, \gamma_2(s), \gamma_3(s))^\top$ be the natural parametrization of the curve (that is a parametrization such that $\dot\gamma_2(s)^2 + \dot\gamma_3(s)^2 = 1$ for any s). Then we can represent $\mathcal{A}_{0,t}$ in the form

$$\mathcal{A}_{0,t} = \left\{(0, \gamma_2(s), \gamma_3(s))^\top + r\mathcal{S}_{(0,\dot\gamma_2(s),\dot\gamma_3(s))^\top},\ s \in [0, L(t))\right\},$$

and, finally, introducing the function

$$Q(s,\varphi,\theta) := \begin{pmatrix} 0 \\ \gamma_2(s) \\ \gamma_3(s) \end{pmatrix} + r \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dot\gamma_2(s) & -\dot\gamma_3(s) \\ 0 & \dot\gamma_3(s) & \dot\gamma_2(s) \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi \\ \cos\theta \\ \sin\theta\sin\varphi \end{pmatrix},$$

we have

$$\mathcal{A}_{0,t} = Q\big([0, L(t)), [0, 2\pi), [0, \pi/2]\big).$$

By simple, but tedious calculations, it follows from (3) that for any $l > 0$ the mapping $Q : [0, l) \times [0, 2\pi) \times [0, \pi/2] \to \mathbb{R}^3$ is bijective. Further, using the equalities $\dot\gamma_2(s)^2 + \dot\gamma_3(s)^2 = 1$ and $\dot\gamma_2(s)\ddot\gamma_2(s) + \dot\gamma_3(s)\ddot\gamma_3(s) = 0$ (the second inequality is the derivative of the first one), the reader can easily prove that for the Jacobian of Q it is true that $\operatorname{Jac} Q(s,\varphi,\theta) = r^2 \sin\theta\cos\theta$. We finish the proof of (4) integrating the Jacobian over the set $[0, L(t)) \times [0, 2\pi) \times [0, \pi/2]$.

There exists a very convenient representation of τ_0. Let η be an exponential r.v. with mean λ; recall that $F : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ is defined as the solution of the equation

$$\int_0^{F(v,t)} |v + as|ds = t.$$

Since $F(v, \cdot)$ is monotone and $F\left(v, \int_0^t |v + as|ds\right) = t$, we have

$$\tau_0 \stackrel{d}{=} F(v_0, \eta). \tag{6}$$

2.2. **The distribution of ν_1.** Recall that at a collision the particle's speed $V_1 = v_0 + a\tau_0$ changes to $V_1 - (1+\alpha)(V_1 \cdot \nu_1)\nu_1$, where ν_1 is the inner unit normal to the first obstacle at the point of collision. To simplify the notations, let us write ν instead of ν_1. As in the previous subsection, we assume that $v_0^\perp = (0, |v_0^\perp|, 0)^\top$.

At first notice that $\nu \in \mathcal{S}_{V_1}$; this vector could be defined by its spherical coordinates $(\varphi_\nu, \theta_\nu^{V_1})$, where the longitude $\varphi_\nu \in [0, 2\pi)$ is the angle between $(1,0,0)^\top$ and ν, and the latitude $\theta_\nu^{V_1} \in [0, \pi/2]$ is the angle between V_1 and ν_1. For any $0 \le t_1 < t_2$, $\varphi \in [0, 2\pi]$, and $\theta \in [0, \pi/2]$, define

$$\mathcal{A}_{t_1,t_2,\varphi,\theta} := \left\{ v_0 s + \frac{as^2}{2} + r\big\{u \in \mathcal{S}_{v_0+as} : \varphi_u < \varphi, \theta_u^{v_0+as} < \theta\big\},\ s \in [t_1, t_2) \right\} \subset \mathcal{A}_{t_1,t_2}.$$

Then

$$\begin{aligned}
\mathbb{P}\Big\{\varphi_\nu < \varphi, \theta_\nu^{V_1} < \theta \,\Big|\, \tau_0 \in [t, t+dt)\Big\} &= \frac{\mathbb{P}\Big\{\varphi_\nu < \varphi, \theta_\nu^{V_1} < \theta, \tau_0 \in [t, t+dt)\Big\}}{\mathbb{P}\big\{\tau_0 \in [t, t+dt)\big\}} \\
&= \frac{\mathbb{P}\big\{\Pi(\mathcal{A}_{0,t}) = 0, \Pi(\mathcal{A}_{t,t+dt,\varphi,\theta}) = 1\big\}}{\mathbb{P}\big\{\Pi(\mathcal{A}_{0,t}) = 0, \Pi(\mathcal{A}_{t,t+dt}) = 1\big\}} + o(1), \quad dt \to 0
\end{aligned}$$

(the term $o(1)$ appears, because since $\lambda_3(\mathcal{A}_{t,t+dt}) = O(dt)$, we have $\mathbb{P}\big\{\Pi(\mathcal{A}_{t,t+dt}) \ge 2\big\} = o(dt)$). The mapping Q from the previous subsection is bijective, therefore the sets $\mathcal{A}_{0,t}$ and $\mathcal{A}_{t,t+dt}$ are disjoint. Consequently,

$$\mathbb{P}\Big\{\varphi_\nu < \varphi, \theta_\nu^{V_1} < \theta \,\Big|\, \tau_0 = t\Big\} = \lim_{dt \to 0} \frac{\mathbb{P}\big\{\Pi(\mathcal{A}_{t,t+dt,\varphi,\theta}) = 1\big\}}{\mathbb{P}\big\{\Pi(\mathcal{A}_{t,t+dt}) = 1\big\}} = \lim_{dt \to 0} \frac{\lambda_3(\mathcal{A}_{t,t+dt,\varphi,\theta})}{\lambda_3(\mathcal{A}_{t,t+dt})};$$

but the numerator is the integral of $\big|\mathrm{Jac}\, Q\big|$ over $[L(t), L(t+dt)) \times [0, \varphi) \times [0, \theta]$, the denominator is the integral of $\big|\mathrm{Jac}\, Q\big|$ over $[L(t), L(t+dt)) \times [0, 2\pi) \times [0, \pi/2]$, and we get

$$\mathbb{P}\Big\{\varphi_\nu < \varphi, \theta_\nu^{V_1} < \theta \,\Big|\, \tau_0 = t\Big\} = \frac{\varphi \sin^2\theta}{2\pi}.$$

We see that the distribution of ν_1 is invariant under rotations around V_1, and

$$\mathbb{P}\big\{\theta_{\nu_1}^{V_1} < \theta\big\} = \sin^2\theta;$$

moreover, $\theta_{\nu_1}^{V_1}$ and τ_0 are independent.

Let us find a suitable representation of ν_1. Suppose that σ is uniformly distributed on the unit sphere S^2 and is independent of V_1 (and of τ_0); let $\tilde{\nu}$ be the unit vector directed along the bisectrix of the angle between σ and V_1. Then, for any fixed V_1, the conditional distributions of ν_1 and $\tilde{\nu}$ coincide. Indeed, both of them are invariant under rotations around V_1, and $\mathbb{P}\big\{\theta_{\tilde{\nu}}^{V_1} < \theta\big\} = \mathbb{P}\big\{\theta_\sigma^{V_1} < 2\theta\big\} = (1 - \cos 2\theta)/2 = \sin^2\theta$. Finally, from $(V_1, \tilde{\nu})\tilde{\nu} = \frac{1}{2}(V_1 + |V_1|\sigma)$ it follows that

$$V_1 - (1+\alpha)(V_1 \cdot \nu_1)\nu_1 \stackrel{d}{=} V_1 - \frac{1+\alpha}{2}\big(V_1 + |V_1|\sigma\big).$$

2.3. **The simplified model.** We simplify the model by stating that we can apply the results of two previous subsections to describe the particle's motion after *each* collision (i.e., we can simply replace v_0 by the speed after the collision). Thus we obtain the model defined by (1) and (2).

The main disparity between the models was discussed in Subsection 1.2. In addition, the models differ, because the distributions of τ_0 and ν_1 were derived under the technical condition (3); clearly, the variables $V_n - \frac{1+\alpha}{2}(V_n + |V_n|\sigma_n)$ and v_0 do not have to satisfy it.

However, our simplifications look quite natural. Indeed, the "disappearance" of obstacles after collisions in a sense means that a particle never returns to already met obstacles. This is reasonable if the obstacles are rare, because there is a drift in the direction of a. It is also sensible to neglect (3) if r or $|a|$ is small; note that since distribution of V_n converges to π_V, the variables $\big(V_n - \frac{1+\alpha}{2}(V_n + |V_n|\sigma_n)\big)^\perp$ converge to a nondegenerate limit.

3. Starting the proof of Theorem 1

3.1. **The position of the particle at the nth collision.** In this subsection we shall find a suitable representation for $X_n := X(t_n)$, where $t_n := \sum_{i=0}^{n-1} \tau_i$ is the moment of the nth collision; additionally, put $t_0 := 0$.

Consider the new Markov chain

$$\Phi_n := \begin{pmatrix} V_n \\ \sigma_n \end{pmatrix}, \qquad n \in \mathbb{N}; \qquad \Phi_0 := \begin{pmatrix} v_0 \\ -v_0/|v_0| \end{pmatrix}.$$

It will be obvious later that the current initial condition describes the particle with initial speed v_0 (at time zero happens the dummy collision, which does not change the speed). Note that V_n and σ_n are independent. Further, introducing the notations

$$\widehat{x} := v - \frac{1+\alpha}{2}\big(v + |v|\sigma\big), \qquad x = \begin{pmatrix} v \\ \sigma \end{pmatrix} \in X := \mathbb{R}^3 \times S^2,$$

we rewrite (1) and (2) as

$$V_{n+1} = \widehat{\Phi_n} + aF(\widehat{\Phi_n}, \eta_n).$$

Thus for the new chain it is true that

$$\Phi_{n+1} = \begin{pmatrix} \widehat{\Phi_n} + aF(\widehat{\Phi_n}, \eta_n) \\ \sigma_{n+1} \end{pmatrix}, \qquad n \geq 0.$$

Let us agree to write coordinates of vectors of $\mathbb{R}^3$ using superscripts; recall that $a^3 = |a|$. From the trivial equalities $V^1_{n+1} = \widehat{\Phi_n}^1$, $V^2_{n+1} = \widehat{\Phi_n}^2$, and $V^3_{n+1} = \widehat{\Phi_n}^3 + |a|\tau_n$ it follows that

$$X_{n+1} = X_n + \begin{pmatrix} \widehat{\Phi_n}^1 \tau_n \\ \widehat{\Phi_n}^2 \tau_n \\ ((V^3_{n+1})^2 - (\widehat{\Phi_n}^3)^2)/(2|a|) \end{pmatrix} = X_n + \frac{1}{|a|} \begin{pmatrix} V^1_{n+1} V^3_{n+1} - \widehat{\Phi_n}^1 \widehat{\Phi_n}^3 \\ V^2_{n+1} V^3_{n+1} - \widehat{\Phi_n}^2 \widehat{\Phi_n}^3 \\ ((V^3_{n+1})^2 - (\widehat{\Phi_n}^3)^2)/2 \end{pmatrix},$$

whence

$$X_{n+1} = \frac{1}{|a|}\begin{pmatrix} \widehat{\Phi_{n+1}}^1\widehat{\Phi_{n+1}}^3 - \widehat{\Phi_0}^1\widehat{\Phi_0}^3 \\ \widehat{\Phi_{n+1}}^2\widehat{\Phi_{n+1}}^3 - \widehat{\Phi_0}^2\widehat{\Phi_0}^3 \\ ((\widehat{\Phi_{n+1}}^3)^2 - (\widehat{\Phi_0}^3)^2)/2 \end{pmatrix} + \frac{1}{|a|}\sum_{i=1}^{n+1}\begin{pmatrix} V_i^1V_i^3 - \widehat{\Phi}_i^1\widehat{\Phi}_i^3 \\ V_i^2V_i^3 - \widehat{\Phi}_i^2\widehat{\Phi}_i^3 \\ ((V_i^3)^2 - (\widehat{\Phi}_i^3)^2)/2 \end{pmatrix}. \tag{7}$$

Besides,

$$t_{n+1} = \sum_{i=0}^{n}\tau_i = \frac{1}{|a|}\sum_{i=0}^{n} V_{i+1}^3 - \widehat{\Phi}_i^3 = \frac{1}{|a|}\big(\widehat{\Phi_{n+1}}^3 - \widehat{\Phi_0}^3\big) + \frac{1}{|a|}\sum_{i=1}^{n+1} V_i^3 - \widehat{\Phi}_i^3. \tag{8}$$

Finally, denoting

$$f(x) := \frac{1}{|a|}\begin{pmatrix} v^1v^3 - \widehat{x}^1\widehat{x}^3 \\ v^2v^3 - \widehat{x}^2\widehat{x}^3 \\ ((v^3)^2 - (\widehat{x}^3)^2)/2 \end{pmatrix}, \quad h(x) := \begin{pmatrix} 0 \\ 0 \\ v^3 - \widehat{x}^3 \end{pmatrix}, \qquad x = \binom{v}{\sigma} \in X,$$

from (7) and (8) we get

$$X_n - c_1 a t_n = \frac{1}{|a|}\begin{pmatrix} \widehat{\Phi_n}^1\widehat{\Phi_n}^3 - \widehat{\Phi_0}^1\widehat{\Phi_0}^3 \\ \widehat{\Phi_n}^2\widehat{\Phi_n}^3 - \widehat{\Phi_0}^2\widehat{\Phi_0}^3 \\ ((\widehat{\Phi_n}^3)^2 - (\widehat{\Phi_0}^3)^2)/2 \end{pmatrix} - \frac{c_1}{|a|}a\big(\widehat{\Phi_n}^3 - \widehat{\Phi_0}^3\big) + \sum_{i=1}^{n}[f - c_1 h](\Phi_i). \tag{9}$$

By definition, put $g := f - c_1 h$ (the value of c_1 will be defined later).

3.2. **The problem in terms of the Markov chain Φ_n.** Between collisions the particle moves with constant acceleration, thus the process $X(t)$ is defined by its values X_n at the points t_n. In fact, we can find the values of $X^1(t)$ and $X^2(t)$ by linear interpolation and the values of $X^3(t)$ by quadratic interpolation with the leading coefficient $|a|/2$. Analogously, the process $Y_t(s)$ is defined by its values at the points t_n/t, but for $Y_t^3(s)$ we shall use quadratic interpolation with the leading coefficient $|a|t^{3/2}/2$.

Let $\widetilde{Y}_t(s)$ be the following process: at the points t_n/t put

$$\widetilde{Y}_t\Big(\frac{t_n}{t}\Big) := Y_t\Big(\frac{t_n}{t}\Big) = \frac{X_n - c_1 a t_n}{\sqrt{t}}, \qquad n \geq 0$$

and define the values at other points via linear interpolation. The only difference between $Y_t(s)$ and $\widetilde{Y}_t(s)$ is in the method of interpolation for the third coordinate. It is easy to see that $Y_t^3(s) - \widetilde{Y}_t^3(s) = |a|t^{3/2}(s - t_n)(s - t_{n+1})/2$ for $s \in [t_n, t_{n+1}]$. Thus, denoting by $n(t)$ the (random) number of collisions by the time t, for the norm $\|\cdot\|_{\mathcal{C}}$ of the space $\mathcal{C}[0,1] = \mathcal{C}\big([0,1],\mathbb{R}^3\big)$ we have

$$\|Y_t(\cdot) - \widetilde{Y}_t(\cdot)\|_{\mathcal{C}} \leq \max_{0\leq k\leq n(t)}\sup_{t_k\leq s\leq t_{k+1}}|Y_t(s) - \widetilde{Y}_t(s)| = \frac{|a|}{8\sqrt{t}}\max_{0\leq k\leq n(t)}\tau_k^2. \tag{10}$$

Then, we introduce the process $Z_t(s)$, putting at the points t_n/t

$$Z_t\Big(\frac{t_n}{t}\Big) := \frac{1}{\sqrt{t}}\sum_{i=1}^{n} g(\Phi_i), \qquad n \geq 0 \tag{11}$$

and defining the values at other points via linear interpolation. Trajectories of $Z_t(s)$ and $\widetilde{Y}_t(s)$ are piecewise linear and their points of interpolation have the same x-coordinates (namely, t_n/t), therefore from (9) we have

$$\|Z_t(\cdot) - \widetilde{Y}_t(\cdot)\|_{\mathcal{C}} \le \frac{1}{\sqrt{t}} \max_{1\le k\le n(t)+1} \left\{ \frac{2.5}{|a|}\big(|\widehat{\Phi_k}|^2 + |\widehat{\Phi_0}|^2\big) + c_1\big(|\widehat{\Phi_k}| + |\widehat{\Phi_0}|\big) \right\}. \tag{12}$$

From (10) and (12) we see that for proving Theorem 1 it is sufficient to check that for all initial conditions $\Phi_0 = x \in X$, it is true that

$$\frac{1}{\sqrt{t}} \max_{0\le k\le n(t)} \tau_k^2 \xrightarrow{\mathbb{P}_x} 0, \qquad \frac{1}{\sqrt{t}} \max_{1\le k\le n(t)+1} |\widehat{\Phi_k}|^2 \xrightarrow{\mathbb{P}_x} 0, \qquad t\to\infty, \tag{13}$$

and there exist constants $c_1 > 0$ and $c_2, c_3 \ge 0$ such that in the space $\mathcal{C}[0,1]$

$$Z_t(\cdot) \xrightarrow{d} Y(\cdot), \qquad t\to\infty. \tag{14}$$

Thus we must study the properties of the Markov chain Φ_n in detail. The necessary facts from the Markov chain theory are stated in Section 4. In Section 5 we prove that the chain Φ_n possess some useful properties. These properties are used in Section 6 for proving (13) and (14). The assertion (14) is the most difficult; we prove it applying the FCLT to the sequence $g(\Phi_n)$.

The problem is reduced to studying properties of the Markov chain Φ_n.

4. Basic facts on Markov chains

The purpose of this section is to describe conditions under which a Markov chain satisfies the law of large numbers (LLN) and the FCLT. We also give a simple method for checking this conditions. All the statements and definitions are taken from [10]; in this section multiple references to this source are omitted.

We begin with several notations. Consider a Markov chain Φ_n, with an arbitrary state space X equipped with a locally compact, separable, metrizable topology and Borel σ-field $\mathcal{B}(X)$. Let $P(x,\cdot)$ be the transition function of Φ_n, let $P^n(x,\cdot)$ be the n-step transition function, and let π be the invariant measure of the chain (in the considered situations, there exists a unique invariant probability measure). Calculating expectations and probabilities, we indicate the initial distribution of the chain, i.e., $\mathcal{L}(\Phi_0)$, with subscripts. For example, $\mathbb{P}_x\big\{\Phi_n \in A\big\}$ imply that $\mathcal{L}(\Phi_0) = \delta_x$ and $\mathbb{E}_\pi \Phi_n$ imply that $\mathcal{L}(\Phi_0) = \pi$. All the considered functions are assumed to be measurable. Finally, by P denote the transition operator; recall that by definition $(Pf)(x) = \int_X f(y)P(x,dy)$, for any functional $f : X \to \mathbb{R}$.

4.1. **Definitions.** A Markov chain is called *irreducible* if there exists a nonzero measure μ on $\mathcal{B}(X)$ such that

$$\mu(A) > 0 \quad\Longrightarrow\quad \mathbb{P}_x\big\{\exists\, n\in\mathbb{N} : \Phi_n \in A\big\} > 0, \qquad x\in X, A\in\mathcal{B}(X);$$

any measure satisfying this condition is called *irreducible measure* of the chain.

An irreducible chain is called *aperiodic* if there does not exist a $d \ge 2$ and there do not exist disjoint sets $E_1,\dots,E_d \in \mathcal{B}(X)$ such that

1) for all $x\in E_d$, $P(x,E_1) = 1$, and for all $x \in E_i$, $P(x,E_{i+1}) = 1$, $\quad i = 1,\dots,d-1$;

2) $\mu\Big(X \setminus \bigcup_{i=1}^{d} E_i\Big) = 0$ holds for every irreducible measure μ of the chain (we modified the definition from [10] using Proposition 4.2.2 and Theorem 5.4.4).

We say that a chain is *(weak) Feller* if the function $P(\,\cdot\,, A)$ is lower semicontinuous for any open set $A \subset X$.

Let μ be a signed measure on $\mathcal{B}(X)$, and let $f : X \to [0, \infty)$ be a functional. We define the *f-norm* of μ as

$$\|\mu\|_f := \sup_{g:|g|\le f} \int_X g d\mu = \int_X f d|\mu|$$

(the inequality $|g| \le f$ is pointwise). The 1-norm is called the *total variation norm*; the notation $\|\mu\|_1$ is replaced by $\|\mu\|$.

Let P_1 and P_2 be Markov transition functions, and let $U : X \to [1, \infty)$ be a functional. By definition, put

$$|||P_1 - P_2|||_U := \sup_{x\in X} \frac{\|P_1(x, \cdot) - P_2(x, \cdot)\|_U}{U(x)}.$$

A Markov chain Φ_n is *ergodic* if there exists a measure π such that for any $x \in X$ it is true that $\|P^n(x, \cdot) - \pi\| \to 0$, as $n \to \infty$; this yields that π is a unique invariant probability measure. A Markov chain is *U-uniformly ergodic* if there exists a measure π such that $|||P^n - \pi|||_U \to 0$, as $n \to \infty$ (we formally put $\pi(x, \cdot) := \pi(\cdot)$). Note that if a chain is U-uniformly ergodic, then it is cU-uniformly ergodic, for any $c > 1$. For irreducible aperiodic chains the 1-uniform ergodicity is equivalent to the well-known Doeblin condition (see Theorem 16.2.3).

Let $g : X \to \mathbb{R}$ be such that $g \in L^1(\pi) = L^1(X, \mathcal{B}(X), \pi)$. The functional equation (in unknown $\bar{g}$)

$$\bar{g} - P\bar{g} = g - \int_X g d\pi \tag{15}$$

is called the *Poisson equation*.

4.2. **Theorems.**

Theorem 2. *Let Φ_n be an ergodic Markov chain, and let $g \in L^1(\pi)$. Then for any initial condition $\Phi_0 = x \in X$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g(\Phi_i) = \int_X g d\pi, \qquad \mathbb{P}_x\text{-}a.s.$$

Proof. Follows from Theorem 17.0.1. □

Theorem 3. *Let Φ_n be an irreducible, aperiodic, Feller Markov chain, and let $Int\,(supp\,\mu) \ne \varnothing$ for some irreducible measure μ of Φ_n. Suppose that the Foster-Lyapunov condition holds: there exist a functional $U : X \to [1, \infty)$, a compact set $C \subset X$, and constants $\beta, b > 0$ such that*

$$PU(x) - U(x) \le -\beta U(x) + b\mathbf{1}_C(x), \qquad x \in X. \tag{16}$$

Then Φ_n is U-uniformly ergodic; moreover, $U \in L^1(\pi)$.

Proof. In view of Proposition 5.5.3 and Theorem 6.0.1, the first statement follows from Theorems 15.0.1 and 16.0.1; the last one is proved in Theorem 14.0.1. □

Theorem 4. *Let Φ_n be a U-uniformly ergodic Markov chain, and let a functional $g : X \to \mathbb{R}$ be such that $g^2 \le U$. Then there exists a solution $\bar g$ of the Poisson equation (15); $\bar g \in L^2(\pi)$; and the constant*

$$\gamma_g^2 := \int_X \left(\bar g^2 - (P\bar g)^2\right) d\pi \ge 0 \tag{17}$$

is well defined. If $\int_X g d\pi = 0$, then for any initial condition $\Phi_0 = x \in X$, in the space $\mathcal{C}[0,1]$

$$S_t(s) := \frac{\sum_{i=1}^{[st]} g(\Phi_i) + (st-[st])g(\Phi_{[st]+1})}{\sqrt{t}} \xrightarrow{d} \sqrt{\gamma_g^2} W(\cdot), \qquad t \to \infty, \tag{18}$$

where W is a Wiener process.

Proof. The existence of a solution of the Poisson equation easily follows from Theorem 17.4.2. The well-posedness of the definition of γ_g^2 (i.e., independence of the choice of the Poisson equation's solution $\bar g$) follows from Proposition 17.4.1. The Cauchy-Bunyakovskii-Schwarz inequality implies that $\gamma_g^2 \ge 0$. Finally, the last statement is the combination of Theorems 17.4.4 and 17.5.4. Although in [10] the processes $S_t(s)$ are defined for positive integer t, in (18) the convergence over $t \in \mathbb{R}$ simply follows from the convergence over $t \in \mathbb{N}$. □

5. Studying properties of the Markov chain Φ_n

We shall frequently use the following trivial inequalities: for any $x = \binom{v}{\sigma} \in X$ it is true that

$$\alpha|v| \le |\widehat{x}| \le |v|;$$

recall that $\alpha \in (0,1)$.

5.1. **Irreducibility, aperiodicity, and Feller property.** First, let us prove that for any $\Phi_0 = x$ the values of $V_2 = \widehat{\Phi_1} + aF(\widehat{\Phi_1}, \eta_1)$ run over the whole $\mathbb{R}^3$. It is sufficient to prove that $\widehat{\Phi_1}$ runs through $\mathbb{R}^3 \setminus B(0, |\widehat{x}|) = \{u \in \mathbb{R}^3 : |u| \ge |\widehat{x}|\}$, because for any fixed $\widehat{\Phi_1}$ the values of $F(\widehat{\Phi_1}, \eta_1)$ run through $\mathbb{R}_+$. We certainly assume that η_i and σ_i run over the whole $\mathbb{R}_+$ and S^2 respectively.

At a collision, the speed $v \in \mathbb{R}^3$ changes to $\tilde v := v - (1+\alpha)(v,\nu)\nu$ (we temporary use the old representation). The reader will easily check that the inverse transformation is $v = \tilde v - (1+\alpha^{-1})(\tilde v, \nu)\nu$, where ν is the same as in the direct transformation. The values of ν run through $\mathcal{S}_v = \{u \in \mathbb{R}^3 | u| = 1, (v,u) \ge 0\}$; consequently, we have $(\tilde v, \nu) \le 0$, that is, $\nu \in \mathcal{S}_{-\tilde v}$. Thus the speed *after* a collision could be equal to a $v \in \mathbb{R}^3$ iff the speed *before* this collision is contained in the set $v^{-1} := \{u \in \mathbb{R}^3 : u = v - (1+\alpha^{-1})(v,\nu)\nu,\ \nu \in \mathcal{S}_{-v}\}$. As above, define σ as a unit vector such that ν is directed along the bisectrix of the angle between σ and $-v$. Then σ runs over S^2, whence $v^{-1} = \{u \in \mathbb{R}^3 : u = v + \frac{1+\alpha^{-1}}{2}(|v|\sigma - v),\ \sigma \in S^2\}$.

We shall prove that $\widehat{\Phi_1}$ runs over $\mathbb{R}^3 \setminus B(0, |\widehat{x}|)$; recall that $\Phi_1 = \binom{\widehat{x} + aF(\widehat{x}, \eta_0)}{\sigma_1}$. Let us show that for any $v \in \mathbb{R}^3 \setminus B(0, |\widehat{x}|)$ the set $v^{-1} \bigcap \{\widehat{x} + as,\ s \ge 0\}$ is nonempty. This is equivalent to the existence of a $s \geq 0$ such that $\left|\widehat{x} + as - \frac{1-\alpha^{-1}}{2}v\right| = \frac{1+\alpha^{-1}}{2}|v|$. But the left-hand side continuously depends on s and increases for large s. Since $|v| \geq |\widehat{x}|$, the value of the left-hand side at $s = 0$ is not greater than $\frac{1+\alpha^{-1}}{2}|v|$. Therefore the required s exists.

Moreover, the distribution of $\widehat{\Phi_1}$ has a density, and this density is positive on $\mathbb{R}^3 \setminus B(0, |\widehat{x}|)$. This fact is intuitively clear, because $F(\widehat{x}, \eta_0)$ has a positive density on $\mathbb{R}_+$ (see (5) and (6)) and σ_1 has a positive density on S^2. The formal proof, whose main part is to calculate the Jacobian of the appropriate transformation, is omitted.

Now it is obvious that the distribution of $V_2 = \widehat{\Phi_1} + aF(\widehat{\Phi_1}, \eta_1)$ has a positive density on $\mathbb{R}^3$. That is why the chain Φ_n is irreducible and $\lambda_3 \otimes U_{S^2}$ is an irreducible measure; here λ_3 is the Lebesgue measure on $\mathbb{R}^3$ and U_{S^2} is the uniform distribution on S^2. Indeed, $\Phi_2 = \binom{V_2}{\sigma_2}$, and the variables V_2 and σ_2 are independent.

We just proved that for any initial condition $\Phi_0 = x$ the distribution of Φ_2, i.e., $P^2(x, \cdot)$, has positive density with respect to $\lambda_3 \otimes U_{S^2}$. Hence for *any* $x \in X$ the measures $P^2(x, \cdot)$ and $\lambda_3 \otimes U_{S^2}$ are equivalent. Thus, by simple arguments, the chain Φ_n is aperiodic.

To prove that Φ_n is a Feller chain, it is sufficient to check that for any open set $A \subset \mathbb{R}^3$ the function $P(\,\cdot\,, A \times S^2)$ is lower semicontinuous. Indeed, from (5) it follows that

$$P(x, A \times S^2) = \int_0^\infty \mathbf{1}_A(\widehat{x} + at)\lambda^{-1}|\widehat{x} + at|e^{-\lambda^{-1}\int_0^t |\widehat{x}+as|ds}dt,$$

and applying the Fatou lemma, we obtain lower semicontinuity.

5.2. U-uniform ergodicity.

Lemma 1. *Let η be an exponential r.v. with mean λ; then for any $c \in \mathbb{R}$*

$$\sup_{v \in \mathbb{R}^3} \mathbb{E}e^{cF(v,\eta)} < \infty.$$

Corollary. *For any initial condition $\Phi_0 = x \in X$, the variables τ_n have exponential moments of any order. Moreover,*

$$\sup_n \mathbb{E}_x e^{c\tau_n} < \infty. \tag{19}$$

Proof of Corollary. Since $\tau_n = F(\widehat{\Phi_n}, \eta_n)$ and $\widehat{\Phi_n}$ is independent of η_n, the proof is obvious. □

Proof of Lemma 1. We only consider the nontrivial case $c > 0$. Take an $s > 0$ and a $v \in \mathbb{R}^3$. Since $\left|-a|v|/|a| + as\right| = \left||a|s - |v|\right| \leq |v + as|$, we see that for any $t > 0$

$$\int_0^t \left|-a|v|/|a| + as\right|ds \leq \int_0^t |v + as|ds.$$

Then, using the definition of F, we have $F(v,\cdot) \le F(-a|v|/|a|,\cdot)$ and therefore $\mathbb{P}\big\{F(v,\eta) > t\big\} \le \mathbb{P}\big\{F(-a|v|/|a|,\eta) > t\big\}$. The right-hand side could be easily calculated, and

$$\begin{aligned}
\mathbb{E}e^{cF(v,\eta)} &= -\int_0^\infty e^{ct} d\,\mathbb{P}\{F(v,\eta) > t\} \\
&= c\int_0^\infty e^{ct}\mathbb{P}\{F(v,\eta) > t\}dt - 1 \\
&\le c\int_0^\infty e^{ct}\mathbb{P}\{F(-a|v|/|a|,\eta) > t\}dt - 1 \\
&= c\int_0^\infty e^{ct} e^{-\lambda^{-1}\int_0^t ||a|s-|v||ds} dt - 1 \\
&= c\int_0^{|v|/|a|} e^{ct-\lambda^{-1}(|v|t-|a|t^2/2)}dt + c\int_{|v|/|a|}^\infty e^{ct-\lambda^{-1}(|a|t^2/2-|v|t+|v|^2/|a|)}dt - 1.
\end{aligned}$$

We now estimate the integrals. For the first one, use the following inequality: if $|v| \ge 4\lambda c$, then $ct - \lambda^{-1}(|v|t - |a|t^2/2) \le -ct$, for all $t \in [0, |v|/|a|]$. Therefore if $|v| \ge 4\lambda c$, then

$$\int_0^{|v|/|a|} e^{ct-\lambda^{-1}(|v|t-|a|t^2/2)}dt \le \int_0^{|v|/|a|} e^{-ct}dt < \int_0^\infty e^{-ct}dt < \infty$$

and this bound does not depend on v. If $|v| < 4\lambda c$, then

$$\int_0^{|v|/|a|} e^{ct-\lambda^{-1}(|v|t-|a|t^2/2)}dt < \int_0^{4\lambda c/|a|} e^{ct+\lambda^{-1}|a|t^2/2}dt < \infty.$$

Estimating the second integral, we apply the following: if $|v| \ge 5\lambda c$, then $ct-\lambda^{-1}(|a|t^2/2-|v|t + |v|^2/|a|) \le -ct$. Thus if $|v| \ge 5\lambda c$, then

$$\int_{|v|/|a|}^\infty e^{ct-\lambda^{-1}(|a|t^2/2-|v|t+|v|^2/|a|)}dt \le \int_{|v|/|a|}^\infty e^{-ct}dt < \int_0^\infty e^{-ct}dt < \infty.$$

If $|v| < 5\lambda c$, then we have

$$\int_{|v|/|a|}^\infty e^{ct-\lambda^{-1}(|a|t^2/2-|v|t+|v|^2/|a|)}dt < \int_0^\infty e^{6ct-\lambda^{-1}|a|t^2/2}dt < \infty.$$

□

To prove the U-uniform ergodicity of Φ_n, we apply Theorem 3. Take a $c > 0$ and check that the Foster-Lyapunov condition (16) holds for $U(x) := e^{c|\widehat{x}|}$.

For the transition operator,

$$PU(x) = \mathbb{E}_x U(\Phi_1) = \mathbb{E}_x e^{c|\widehat{\Phi_1}|} = \mathbb{E}_x e^{c|V_1 - \frac{1+\alpha}{2}(V_1+|V_1|\sigma_1)|}.$$

We define the function

$$\gamma(|v|) := \int_{S^2} e^{c\left|v-\frac{1+\alpha}{2}(v+|v|\zeta)\right|-c|v|}dU_{S^2}(\zeta), \qquad v \in \mathbb{R}^3,$$

which is obviously monotone and $\gamma(|v|) \to 0$ as $|v| \to \infty$. Because σ_1 is independent of V_1 and of Φ_0,

$$\begin{aligned} PU(x) &= \mathbb{E}_x \gamma(|V_1|) e^{c|V_1|} \\ &\le \mathbb{E}_x \gamma(|V_1|) e^{c|\widehat{\Phi_0}|} e^{c|a|\tau_0} \\ &\le \mathbb{E}_x \left(\gamma(|\widehat{\Phi_0}/2|) \mathbf{1}_{\{|a|\tau_0 \le |\widehat{\Phi_0}|/2\}} + \gamma(0) \mathbf{1}_{\{|a|\tau_0 > |\widehat{\Phi_0}|/2\}} \right) e^{c|\widehat{\Phi_0}|} e^{c|a|\tau_0} \\ &\le \left(\gamma(|\widehat{x}/2|) \mathbb{E} e^{c|a|F(\widehat{x},\eta_0)} + \mathbb{E} e^{c|a|F(\widehat{x},\eta_0)} \mathbf{1}_{\{|a|F(\widehat{x},\eta_0) > |\widehat{x}|/2\}} \right) e^{c|\widehat{x}|} \\ &\le \left(\gamma(\alpha|v|/2) \mathbb{E} e^{c|a|F(\widehat{x},\eta_0)} + \mathbb{E} e^{c|a|F(\widehat{x},\eta_0)} \mathbf{1}_{\{|a|F(\widehat{x},\eta_0) > \alpha|v|/2\}} \right) U(x), \end{aligned}$$

where as usual $x = \binom{v}{\sigma}$. It follows from Lemma 1 that the factor of $U(x)$ tends to zero as $|v| \to \infty$, whence for any $\beta \in (0,1)$ there exists an $R > 0$ such that

$$PU(x) - U(x) \le -\beta U(x), \qquad x \notin C_R := B(0,R) \times S^2.$$

Clearly, for some $b > 0$

$$PU(x) - U(x) \le -\beta U(x) + b\mathbf{1}_{C_R}(x), \qquad x \in X.$$

Thus the condition (16) holds and, consequently, for any $c > 0$ the Markov chain Φ_n is $e^{c|\widehat{x}|}$-uniformly ergodic.

5.3. **The invariant measure.** By definition of U-uniform ergodicity, there exists a unique invariant measure π of the chain. Since for every n the measure $P^n(x,\cdot)$ is a product of some measure on $\mathcal{B}(\mathbb{R}^3)$ and U_{S^2}, for the limit we also have

$$\pi = \pi_V \otimes U_{S^2},$$

where π_V is a probability measure on $\mathcal{B}(\mathbb{R}^3)$. By the reasons of symmetry, π_V is invariant under rotations around the third coordinate axis.

Further, we claim that the measure π_V has a density. Indeed, in Subsection 5.1 we proved that $P^2(x,\cdot)$ has a density; moreover, it could be shown that for all $n \ge 2$ the measures $P^n(x,\cdot)$ have densities, that is, $P^n(x,\cdot) \prec \lambda_3 \otimes U_{S^2}$. Thus, passing to the limit, $\pi = \pi_V \otimes U_{S^2} \prec \lambda_3 \otimes U_{S^2}$ and $\pi_V \prec \lambda_3$.

Theorem 3 implies that $e^{c|\widehat{x}|} \in L^1(\pi)$ for any c, and we can prove the following

Proposition 1. *For any initial condition $\Phi_0 = x \in X$, the variables V_n have exponential moments of any order. Moreover,*

$$\sup_n \mathbb{E}_x e^{c|V_n|} < \infty. \tag{20}$$

Proof. We begin with $|V_n| = |\widehat{\Phi_{n-1}} + a\tau_{n-1}| \le |V_{n-1}| + |a|\tau_{n-1} \le \cdots \le |\widehat{x}| + |a| \sum_{i=0}^{n-1} \tau_i$. Then, via the Hölder inequality and Lemma 1,

$$\mathbb{E}_x e^{c|V_n|} \le e^{c|\widehat{x}|} \mathbb{E}_x \prod_{i=0}^{n-1} e^{c|a|\tau_i} \le e^{c|\widehat{x}|} \prod_{i=0}^{n-1} (\mathbb{E}_x e^{c|a|n\tau_i})^{1/n} \le e^{c|\widehat{x}|} \sup_{v \in \mathbb{R}^3} \mathbb{E} e^{c|a|nF(v,\eta)} < \infty,$$

hence the exponential moments exist. To prove (20), we combine the trivial inequality $e^{c|u|} \le e^{c\alpha^{-1}|\widehat{y}|}$, for $y = \binom{u}{\rho} \in X$, and the definition of $e^{c\alpha^{-1}|\widehat{y}|}$-uniform ergodicity of Φ_n:

$$\lim_{n\to\infty} \mathbb{E}_x e^{c|V_n|} = \lim_{n\to\infty} \int_X e^{c|u|} P^n(x, dy) = \int_X e^{c|u|} d\pi(y) \le \int_X e^{c\alpha^{-1}|\widehat{y}|} d\pi(y) < \infty.$$

□

6. Finishing the proof of Theorem 1

Recall that we must prove (13) and (14). Let us start with the following lemmas.

Lemma 2. *For any initial condition $\Phi_0 = x \in X$,*

$$\tau_n = O(\log n), \qquad |V_n| = O(\log n), \qquad \mathbb{P}_x\text{-}a.s.$$

Proof. Applying the Chebyshev inequality and then using (19), we have

$$\mathbb{P}_x\big\{\tau_n > 2\log n\big\} \le \frac{\mathbb{E}_x e^{\tau_n}}{n^2} \le \frac{1}{n^2} \sup_n \mathbb{E}_x e^{c\tau_n}.$$

Thus the first statement immediately follows from the Borel-Cantelli lemma. Similarly, we prove the second statement via (20). □

Lemma 3. *There exists a $c_4 > 0$ such that for any initial condition $\Phi_0 = x \in X$,*

$$\lim_{n\to\infty} \frac{t_n}{n} = c_4, \qquad \mathbb{P}_x\text{-}a.s.$$

Proof. Recalling (8) and the introduced notations, we see that it is sufficient to prove

$$\lim_{n\to\infty} \frac{\widehat{\Phi_n}^3}{n} = 0, \qquad \mathbb{P}_x\text{-}a.s.;$$

the existence of a c_4 such that

$$\lim_{n\to\infty} \frac{1}{n|a|} \sum_{i=1}^{n} h^3(\Phi_i) = c_4, \qquad \mathbb{P}_x\text{-}a.s.;$$

and positiveness of c_4.

Since $|\widehat{\Phi_n}| \le |V_n|$, from Lemma 2 we immediately obtain the first statement. Further, we can apply Theorem 2 to prove the second statement, because $|h^3(x)| = |v^3 - \widehat{x}^3| \le (1 + \alpha^{-1})|\widehat{x}| < (1+\alpha^{-1})e^{|\widehat{x}|} \in L^1(\pi)$ and thus $h^3 \in L^1(\pi)$. By definition, put $c_4 := |a|^{-1} \int_X h^3 d\pi$. The proof of positiveness of c_4, which is quite simple, could be found in [14]. □

6.1. **Proof of** (13)**.** For the r.v. $n(t)$, which denotes the number of collisions by the time t, it is true that

$$\lim_{t\to\infty} n(t) = \infty, \qquad \mathbb{P}_x\text{-}a.s. \tag{21}$$

To prove this, assume the converse. Then we can find a $k \ge 0$ such that with nonzero probability the particle collides with obstacles only k times. Thus the probability of $\tau_k = \infty$ is nonzero that contradicts with the existence of exponential moments.

By Lemma 3 and (21),

$$\lim_{t\to\infty}\frac{t_{n(t)}}{n(t)}=c_4,\qquad \mathbb{P}_x\text{-}a.s.,$$

but since $t_{n(t)}\le t<t_{n(t)+1}=t_{n(t)}+\tau_{n(t)}$, from Lemma 2 we have

$$\lim_{t\to\infty}\frac{t}{n(t)}=c_4,\qquad \mathbb{P}_x\text{-}a.s. \tag{22}$$

Applying Lemma 2 once again (recall that $|\widehat{\Phi_k}|\le|V_k|$), we prove (13) (to be precise, we prove $\mathbb{P}_x$-a.s. convergence, which is much stronger).

6.2. **Definition of the constants** c_1, c_2, c_3**.** We put

$$c_1:=c_4^{-1}|a|^{-1}\int_X f^3 d\pi;$$

for the quite tedious proof of positiveness of c_1, see [14].

Now we can easily check that for $g=f-c_1h$ it is true that $\int_X g d\pi=0$. In fact, for the first and the second coordinates, this follows from simple calculations, where the representation $\pi=\pi_V\otimes U_{S^2}$ and symmetry of π_V are used. For the third coordinate, we apply the equality $c_4=|a|^{-1}\int_X h^3 d\pi$.

Let us define c_2 and c_3. By $|g(x)|\le 2.5|a|^{-1}(1+\alpha^{-1})|\widehat{x}|^2+c_1(1+\alpha^{-1})|\widehat{x}|$, there exists a $c>1$ such that $|g(x)|^2\le ce^{|\widehat{x}|}$. Thus (the chain Φ_n is $ce^{|\widehat{x}|}$-uniformly ergodic) the functionals g^1, g^2, and g^3 satisfy the conditions of Theorem 4, and there exist solutions $\overline{g^1},\overline{g^2},\overline{g^3}\in L^2(\pi)$ of the Poisson equations. Define

$$c_2:=\sqrt{c_4^{-1}\gamma_{g^2}^2},\qquad c_3:=\sqrt{c_4^{-1}\gamma_{g^3}^2},$$

and also $\bar g:=(\overline{g^1},\overline{g^2},\overline{g^3})^\top$,

$$K:=\int_X\left(\bar g\,\bar g^\top-(P\bar g)(P\bar g)^\top\right)d\pi.$$

In [14] we proved (using the axial symmetry of π_V) that the matrix K is diagonal and $\gamma_{g^1}^2=\gamma_{g^2}^2$, thus

$$K=c_4\begin{pmatrix}c_2^2&0&0\\0&c_2^2&0\\0&0&c_3^2\end{pmatrix}.$$

6.3. **Proof of** (14)**.** In this subsection the following proposition plays the key role.

Proposition 2. *For the processes $S_t(s)$, defined in* (18),

$$S_t(\cdot)\xrightarrow{d}\sqrt{c_4}Y(\cdot),\qquad t\to\infty.$$

Proof. It is sufficient to show that for any $u\in\mathbb{R}^3$

$$(S_t(\cdot),u)\xrightarrow{d}\sqrt{c_4}(Y(\cdot),u),\qquad t\to\infty.$$

On the one hand,

$$\sqrt{c_4}(Y(\cdot),u) \stackrel{d}{=} \sqrt{c_4\big((c_2u^1)^2+(c_2u^2)^2+(c_3u^3)^2\big)}W(\cdot)=\sqrt{(Ku,u)}W(\cdot).$$

On the other hand, the functional (g,u) satisfies the conditions of Theorem 4, thus

$$(S_t(s),u)=\frac{\sum_{i=1}^{[st]}(g,u)(\Phi_i)+(st-[st])(g,u)(\Phi_{[st]+1})}{\sqrt{t}} \xrightarrow{d} \sqrt{\gamma^2_{(g,u)}}W(\cdot), \qquad t\to\infty.$$

And since, obviously, $\overline{(g,u)}=(\bar g,u)$, from (17) we get

$$\gamma^2_{(g,u)}=\int_X\big((\bar g,u)^2-(P(\bar g,u))^2\big)\,d\pi=\int_X\big((\bar g,u)^2-(P\bar g,u)^2\big)\,d\pi=(Ku,u).$$

□

Let us put $\widetilde{S}_t(s):=S_t(c_4^{-1}s)$; then $S_t(c_4^{-1}s)=\sqrt{c_4^{-1}}S_{c_4^{-1}t}(s)$, and in the space $\mathcal{C}[0,1]$ (and, moreover, in $\mathcal{C}[0,l]$, for every $l>0$)

$$\widetilde{S}_t(\cdot)\xrightarrow{d}Y(\cdot), \qquad t\to\infty. \tag{23}$$

Therefore we will prove (14) if we show that

$$\|Z_t(\cdot)-\widetilde{S}_t(\cdot)\|_{\mathcal{C}}\xrightarrow{\mathbb{P}_x}0, \qquad t\to\infty. \tag{24}$$

For this purpose, introduce the process $u_t(s)$, putting at the points t_n/t

$$u_t\Big(\frac{t_n}{t}\Big):=\frac{c_4n}{t}, \qquad n\geq 0$$

and defining the values at other points via linear interpolation. Using the definitions of $Z_t(s)$ and $S_t(s)$, i.e., (11) and (18), we have $Z_t\big(t_n/t\big)=S_t\big(n/t\big)$. But $S_t\big(n/t\big)=\widetilde{S}_t\big(c_4n/t\big)=\widetilde{S}_t\big(u_t\big(t_n/t\big)\big)$, whence at the points t_n/t the equality $Z_t\big(t_n/t\big)=\widetilde{S}_t\big(u_t\big(t_n/t\big)\big)$ holds. However,

$$Z_t(s)=\widetilde{S}_t(u_t(s))$$

is true for *every* s ! In fact, trajectories of $Z_t(s)$ and $\widetilde{S}_t(u_t(s))$ are piecewise linear (for the last one, as a composition of piecewise linear functions) and their points of interpolation have the same x-coordinates (namely, t_n/t). As we saw before, the values at these points coincide.

We see that $Z_t(s)$ is obtained from $\widetilde{S}_t(s)$ by the random change of time. Suppose $\|u_t(\cdot)\|_{\mathcal{C}}\leq 2$; then

$$\|Z_t(\cdot)-\widetilde{S}_t(\cdot)\|_{\mathcal{C}}=\|\widetilde{S}_t(u_t(\cdot))-\widetilde{S}_t(\cdot)\|_{\mathcal{C}}\leq\omega_{\widetilde{S}_t\big|_{[0,2]}}\big(\|u_t(\cdot)-\mathrm{id}\|_{\mathcal{C}}\big),$$

where ω is the modulus of continuity, $\widetilde{S}_t\big|_{[0,2]}$ is the restriction of $S_t(s)$ to $[0,2]$. Hence for any $1>\delta>0$ and $\varepsilon>0$

$$\begin{aligned}\mathbb{P}_x\big\{\|Z_t(\cdot)-\widetilde{S}_t(\cdot)\|_{\mathcal{C}}\geq\varepsilon\big\} &\leq \mathbb{P}_x\big\{\|u_t(\cdot)-\mathrm{id}\|_{\mathcal{C}}\geq\delta\big\}+\mathbb{P}_x\big\{\omega_{\widetilde{S}_t\big|_{[0,2]}}(\delta)\geq\varepsilon\big\}\\ &\leq \mathbb{P}_x\big\{\|u_t(\cdot)-\mathrm{id}\|_{\mathcal{C}}\geq\delta\big\}+\sup_{t>0}\mathbb{P}_x\big\{\omega_{\widetilde{S}_t\big|_{[0,2]}}(\delta)\geq\varepsilon\big\}.\end{aligned}$$

Let us proceed to the limit as $t \to \infty$ and then proceed to the limit as $\delta \to 0$. Now it is obvious that (24) holds if for any $1 > \delta > 0$

$$\lim_{t\to\infty} \mathbb{P}_x\big\{\|u_t(\cdot) - \mathrm{id}\|_{\mathcal{C}} \geq \delta\big\} = 0 \tag{25}$$

and for any $\varepsilon > 0$

$$\lim_{\delta\to 0} \sup_{t>0} \mathbb{P}_x\big\{\omega_{\widetilde{S}_t|_{[0,2]}}(\delta) \geq \varepsilon\big\} = 0. \tag{26}$$

At first we prove (25). Writing $u_t(s)$ in the explicit form, we have

$$\|u_t(\cdot) - \mathrm{id}\|_{\mathcal{C}} = \sup_{0\leq s\leq 1}\Big|\frac{c_4 n(st)}{t} + \frac{st - t_{n(st)}}{\tau_{n(st)}} \cdot \frac{c_4}{t} - s\Big| \leq \sup_{0\leq s\leq 1}\Big|\frac{c_4 n(st) - st}{t}\Big| + \frac{c_4}{t};$$

thus, by (22),

$$\lim_{t\to\infty} \|u_t(\cdot) - \mathrm{id}\|_{\mathcal{C}} = 0, \qquad \mathbb{P}_x\text{-}a.s.,$$

which is much stronger than (25).

It remains to check (26) to complete the proof of Theorem 1. The family of probability measures $\{\mathbb{P}_x \circ \widetilde{S}_t|_{[0,2]}^{-1}\}_{t>0}$ on $\mathcal{B}\big(\mathcal{C}[0,2]\big)$ is relatively weakly compact. This follows from (23) and from $\mathbb{P}_x$-a.s. continuity of $\|\widetilde{S}_t\|_{\mathcal{C}[0,2]}$ in $t \in [0,\infty)$. The space $\mathcal{C}[0,2]$ is a Polish space, thus the relatively weakly compact family $\{\mathbb{P}_x \circ \widetilde{S}_t|_{[0,2]}^{-1}\}_{t>0}$ is tight. By the well-known fact (see [2]) about tight families of probability measures on $\mathcal{B}\big(\mathcal{C}[0,2]\big)$, for any $\varepsilon > 0$

$$\lim_{\delta\to 0} \sup_{t>0} \mathbb{P}_x \circ \widetilde{S}_t|_{[0,2]}^{-1}\Big\{p : \omega_p(\delta) \geq \varepsilon\Big\} = \lim_{\delta\to 0} \sup_{t>0} \mathbb{P}_x\Big\{\omega_{\widetilde{S}_t|_{[0,2]}}(\delta) \geq \varepsilon\Big\} = 0.$$

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