ISSN 1063-4541, Vestnik St. Petersburg University. Mathematics, 2008, Vol. 41, No. 4, pp. 360–366. © Allerton Press, Inc., 2008.
Original Russian Text © S.B. Tikhomirov, 2008, published in Vestnik Sankt-Peterburgskogo Universiteta. Seriya 1. Matematika, Mekhanika, Astronomiya, 2008, No. 4, pp. 90–97.

MATHEMATICS

Interiors of Sets of Vector Fields with Shadowing Corresponding to Certain Classes of Reparameterizations

S. B. Tikhomirov

Received May 18, 2008

Abstract—The structure of the $\mathbf{C}^1$-interiors of sets of vector fields with various forms of the shadowing property is studied. The fundamental difference between the problem under consideration and its counterpart for discrete dynamical systems generated by diffeomorphisms is the reparameterization of shadowing orbits. Depending on the type of reparameterization, Lipschitz and oriented shadowing properties are distinguished. As is known, structurally stable vector fields have the Lipschitz shadowing property. Let X be a vector field, and let p and q be its points of rest or closed orbits. Suppose that the stable manifold of p and the unstable manifold of q have a nontransversal intersection point. It is shown that, in this case, the vector field X does not have the Lipschitz shadowing property. If one of the orbits p and q is closed, then X does not have the oriented shadowing property. These assertions imply that the $\mathbf{C}^1$-interior of the set of vector fields with the Lipschitz shadowing property coincides with the set of structurally stable vector fields. If the dimension of the manifold under consideration is at most 3, then a similar result is valid for the oriented shadowing property. We study the structure of the $\mathbf{C}^1$-interiors of sets of vector fields with various forms of the shadowing property. It is shown that, in the case of the Lipschitz shadowing property, it coincides with the set of structurally stable systems. For manifolds of dimension at most 3, a similar result is valid for the oriented shadowing property.

DOI: 10.3103/S1063454108040122

1. INTRODUCTION

The problem of shadowing pseudo-orbits is related to the following question: Under what conditions is any pseudo-orbit of a dynamical system close to a orbit? The study of this question was initiated by Anosov [1] and Bowen [2]. The current state-of-the-art in shadowing theory is reviewed in monographs [3, 4].

The fundamental difference between the shadowing problem for flows and that for discrete dynamical systems generated by diffeomorphisms consists in the reparameterization of shadowing orbits.

The purpose of this paper is to describe the structure of the $\mathbf{C}^1$-interiors of sets of vector fields with certain pseudo-orbit shadowing properties.

2. BASIC NOTATION AND MAIN RESULTS

Let M be a smooth closed (i.e., compact without boundary) n-manifold with Riemannian metric dist. By $\mathcal{F}(M)$ we denote the space of smooth vector fields on M with the $\mathbf{C}^1$-topology. For a vector field $X \in \mathcal{F}(M)$, $\phi(t, x)$ denotes an orbit of X for which $\phi(0, x) = x$.

Definition 1. Let $d > 0$. We define a *d-pseudo-orbit* of the field X as a mapping $g\colon \mathbf{R} \longrightarrow M$ such that $\operatorname{dist}(g(t+\tau), \phi(t, g(\tau))) < d$ for $|t| < 1$ and $\tau \in \mathbf{R}$.

Let us introduce the notion of shadowing for flows. The key role in shadowing for flows is played by reparameterizations.

Definition 2. A *reparameterization* is an increasing homeomorphism $h\colon \mathbf{R} \longrightarrow \mathbf{R}$ for which $h(0) = 0$. For $a > 0$, $\operatorname{Rep}(a)$ denotes the set of reparameterization satisfying the inequality

$$\left|\frac{h(t_1) - h(t_2)}{t_1 - t_2} - 1\right| \le a \quad \text{for} \quad t_1, t_2 \in \mathbf{R}, \quad t_1 \ne t_2.$$

Definition 3. We say that a flow ϕ has the *oriented shadowing property* if, for any $\varepsilon > 0$, there exists a $d > 0$ such that, for any d-pseudo-orbit g, we can find a point p and a reparameterization h satisfying the condition

$$\operatorname{dist}(\phi(h(t), p), g(t)) < \varepsilon, \quad t \in \mathbf{R}.$$

Definition 4. We say that a flow ϕ has the *Lipschitz shadowing property* if there exist L_0, $D_0 > 0$ such that, for any $d < D_0$ and any d-pseudo-orbit g, we can find a point p and a reparameterization $h \in \mathrm{Rep}(L_0 d)$ satisfying the condition

$$\mathrm{dist}(\phi(h(t), p), g(t)) < L_0 d, \quad t \in \mathbf{R}.$$

We denote the sets of vector fields with the oriented and Lipschitz shadowing properties by OrSh and LipSh, respectively. By S we denote the set of structurally stable vector fields, T denotes the set of vector fields whose all points of rest and closed orbits are hyperbolic, and KS is the set of Kupka–Smale fields [5]. Clearly, LipSh $\subset$ OrSh.

For any set $A \subset \mathcal{F}(M)$, let $\mathrm{Int}^1(A)$ denote the $\mathbf{C}^1$-interior of A. For a vector field X, $\mathrm{Per}(X)$ is the set of rest points and closed orbits of X. For any hyperbolic orbit $p \in \mathrm{Per}(X)$, $W^s(p)$ and $W^u(p)$ denote its stable and unstable manifolds, respectively.

It was shown in [6] that $S \subset$ LipSh. Since the set S is $\mathbf{C}^1$-open, it follows that $S \subset \mathrm{Int}^1(\mathrm{LipSh})$. The main results of this paper are as follows.

Theorem 1. $S = \mathrm{Int}^1(\mathrm{LipSh})$.

Theorem 2. If $\dim M \leq 3$, then $S = \mathrm{Int}^1(\mathrm{OrSh})$.

3. PROOF OF THEOREM 1

A modification of the argument used in [7] in the case of diffeomorphisms easily proves the following assertion.

Lemma 1. $\mathrm{Int}^1(\mathrm{OrSh}) \subset T$.

Gan proved in [8] that $\mathrm{Int}^1(KS) = S$. Thus, Theorem 1 is implied by the following assertion.

Lemma 2. Suppose that $X \in \mathrm{Int}^1(\mathrm{LipSh})$ and $p, q \in \mathrm{Per}(X)$. If $r \in W^u(q) \cap W^s(p)$, then r is a transversal intersection point of $W^u(q)$ and $W^s(p)$.

Proof. We give a proof of this lemma for the most difficult case, in which p and q are points of rest. In the other cases, a similar assertion for the oriented shadowing property can be proved by using methods from [7, 9].

Lemma 3. Suppose that $X \in \mathrm{Int}^1(\mathrm{OrSh})$, γ_1 is a closed orbit of X, $\gamma_2 \in \mathrm{Per}(X)$, and $r_0 \in W^s(\gamma_1) \cap W^u(\gamma_2)$. Then, r_0 is a transversal intersection point of $W^s(\gamma_1)$ and $W^u(\gamma_2)$.

We need two elementary technical lemmas, which we state without proof. Consider a flow $\varphi(t, x)$ in the plane $\mathbf{R}^2$ generated by a linear autonomous system of the form

$$\dot{x} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} x, \quad x \in \mathbf{R}^2, \text{ where } a > 0 \text{ and } b \neq 0.$$

For a point $x \in \mathbf{R}^2 \setminus \{0\}$, we denote the point $x/|x| \in S^1$ by $\arg(x)$.

Lemma 4. For any ε, $L > 0$, there exist positive numbers $T = T(\varepsilon, L)$ and $d_0 = d_0(\varepsilon, L)$ such that if

$$\begin{gathered} d < d_0, \quad x_0, x_1 \in \mathbf{R}^2, \quad |x_0| \geq d, \quad h(t) \in \mathrm{Rep}(Ld), \\ |\varphi(t, x_0) - \varphi(h(t), x_1)| < Ld \text{ at } t \in [0, T], \end{gathered} \tag{1}$$

then $|\arg(x_1) - \arg(x_0)| < \varepsilon$. A one-dimensional (and simpler) analogue of Lemma 4 is the following assertion, which refers to the differential equation $\dot{x} = ax$ on the line and its flow $\varphi(t, x) = xe^{at}$.

Lemma 5. For any ε, $L > 0$, there exist positive numbers $T = \mathrm{T}(\varepsilon, L)$ and $d_0 = d_0(\varepsilon, L)$ such that if

$$d < d_0, \quad x_0, x_1 \in \mathbf{R}, \quad |x_0| > d, \quad h(t) \in \mathrm{Rep}(Ld),$$

and inequality (1) holds, then

$$\frac{|x_1 - x_0|}{|x_0|} < \varepsilon.$$

We proceed to prove Lemma 2. Suppose that, on the contrary, r is a point of nontransversal intersection of $W^u(q)$ and $W^s(p)$. It was shown in [7] and [9] that any $\mathbf{C}^1$-neighborhood of the field X contains a field X'

such that p and q are hyperbolic rest points of X', r is a nontransversal intersection point of $W^u(q)$ and $W^s(p)$, and the field X' is linear in some neighborhoods N_p and N_q of the points p and q, respectively. Since $X \in \mathrm{Int}^1(\mathrm{LipSh})$, it follows that X' can be chosen to belong to $\mathrm{Int}^1(\mathrm{LipSh})$ as well. To further simplify the exposition, we denote X' as X and the flow generated by X' as φ. In what follows, in the course of the proof, we perturb the field X in a similar way several times, so that the perturbed field remains in $\mathrm{Int}^1(\mathrm{LipSh})$, and denote the new field by X and the flow by ϕ.

Let us identify the neighborhoods N_p and N_q with the space $\mathbf{R}^n$. In N_p and N_q, we introduce local coordinates (y, z) and (ξ, η) so that p and q are the origins in N_p and N_q, respectively, and the Jacobian matrices in these coordinates (possibly, for a perturbed field X) have the form $DX(p) = \mathrm{diag}(A_p, B_p)$, where $\mathrm{Re}(\lambda_j) < 0$ for the eigenvalues of A_p, $\mathrm{Re}(\lambda_j) > 0$ for the eigenvalues of B_p, and $B_p = \mathrm{diag}(\lambda_1, \dots, \lambda_{u_1}, D_1, \dots, D_{u_2})$, where $\lambda_1, \dots, \lambda_{u_1} \in \mathbf{R}$ and the D_j are 2×2 matrices of the form

$$D_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}, \text{ where } a_j > 0 \text{ and } b_j \neq 0, \quad j \in \{1, \dots, u_2\}.$$

Similarly, $DX(q) = \mathrm{diag}(A_q, B_q)$, where $\mathrm{Re}(\lambda_j) > 0$ for the eigenvalues A_q, $\mathrm{Re}(\lambda_j) < 0$ for the eigenvalues of B_q, and $B_q = \mathrm{diag}(\mu_1, \dots, \mu_{s_1}, \tilde{D}_1, \dots, \tilde{D}_{s_2})$, where $\mu_1, \dots, \mu_{s_1} \in \mathbf{R}$ and the $\tilde{D}_j$ are 2×2 matrices of the form

$$\tilde{D}_j = \begin{pmatrix} \tilde{a}_j & -\tilde{b}_j \\ \tilde{b}_j & \tilde{a}_j \end{pmatrix}, \text{ where } \tilde{a}_j < \text{ and } \tilde{b}_j \neq 0, \quad j \in \{1, \dots, s_2\}.$$

Thus, in the neighborhoods N_p and N_q (in what follows, we assume that the whole consideration is performed on the union of N_p and N_q and a small neighborhood of the orbit of r), we have

$$W^s(p) = \{z = 0\}, \quad W^u(p) = \{y = 0\}, \quad W^s(q) = \{\eta = 0\}, \quad W^u(q) = \{\xi = 0\}.$$

Let us introduce the notations $S_p = W^s(p)$, $U_p = W^u(p)$, $S_q = W^s(q)$, and $U_q = W^u(q)$. Suppose that $S_q = S_q^{(1)} \oplus \dots \oplus S_q^{(l)}$, where $l = s_1 + s_2$ and $S_q^{(1)}, \dots, S_q^{(l)}$ are one- or two-dimensional subspaces invariants with respect to $DX(q)$. Similarly, $U_p = U_p^{(1)} \oplus \dots \oplus U_p^{(m)}$, where $m = u_1 + u_2$ and $U_p^{(1)}, \dots, U_p^{(m)}$ are one- or two-dimensional subspaces invariant with respect to $DX(p)$.

For $j = 1, \dots, l$, let $\Pi_q^{(j)}$ denote the projectors onto $S_q^{(j)}$ parallel to $U_q \oplus S_q^{(1)} \oplus \dots \oplus S_q^{(j-1)} \oplus S_q^{(j+1)} \oplus \dots \oplus S_q^{(l)}$. We have

$$\Pi_q^{(j)} S_q^{(j)} = S_q^{(j)} \text{ and } \Pi_q^{(j)} \Pi_q^{(k)} = 0, \text{ where } j, k = 1, \dots, l, \quad j \neq k.$$

Let Π_q denote the projector onto S_q parallel to U_q; then, $\Pi_q = \Pi_q^{(1)} + \dots + \Pi_q^{(l)}$.

Let $\Pi_p^{(1)}, \dots, \Pi_p^{(m)}$ be the projectors onto $U_p^{(1)}, \dots, U_p^{(m)}$, respectively. Then,

$$\Pi_p^{(i)} U_p^{(i)} = U_p^{(i)} \text{ and } \Pi_p^{(i)} \Pi_p^{(k)} = 0, \text{ where } i, k = 1, \dots, m, \quad i \neq k.$$

By Π_p we denote the projector onto U_p parallel to S_p; we have $\Pi_p = \Pi_p^{(1)} + \dots \Pi_p^{(m)}$.

On the orbit $\phi(t, r)$, choose points $a_p \in N_p$ and $a_q \in N_q$ so that $\phi(t, a_p) \in N_p$ and $\phi(-t, a_q) \in N_q$ for any $t > 0$. For some $\tau > 0$, we have $a_p = \phi(\tau, a_q)$. We set $v_p = X(a_p)$ and $v_q = X(a_q)$. Clearly, $v_p \in S_p$ and $v_q \in U_q$.

Let $\tilde{\Sigma}_p$ be the hyperplane in S_p orthogonal to v_p, and let Σ_p be the affine $(n-1)$-dimensional subspace defined by $\Sigma_p = a_p + \tilde{\Sigma}_p + U_p$. Similarly, $\tilde{\Sigma}_q$ is the hyperplane in U_q orthogonal to v_q and Σ_q is the affine $(n-1)$-dimensional subspace defined by $\Sigma_q = a_q + \tilde{\Sigma}_q + S_q$. Clearly, Σ_p and Σ_q have no contact with the field X in small neighborhoods of the points a_p and a_q. Let $K: \Sigma_q \longrightarrow \Sigma_p$ denote the corresponding Poincaré mapping.

Perturbing the field X and choosing appropriate coordinates near the piece $\phi([o, \tau], a_q)$ of the orbit, we can achieve

• the fulfillment of the equality $K(x) = \phi(\tau, x)$ for $x \in \Sigma_q$ close to a_q;

• the linearity of the mapping K (under the natural identification of Σ_q with $\tilde{\Sigma}_q \oplus S_q$ and Σ_p with $\tilde{\Sigma}_p \oplus U_p$). Clearly, in this case, we have

$$T_{a_p}W^u(q) = K\tilde{\Sigma}_q + v_p \quad \text{and} \quad T_{a_p}W^s(p) = \tilde{\Sigma}_p + v_p. \tag{2}$$

The nontransversality of the intersection of $W^u(q)$ and $W^s(p)$ at the point a_p means that $T_{a_p}W^u(q) + T_{a_p}W^s(p) \neq \mathbf{R}^n$. By virtue of relations (2), this means that $v_p + \tilde{\Sigma}_p + K\tilde{\Sigma}_q \neq \mathbf{R}^n$. Applying the equality $v_p + \tilde{\Sigma}_p = S_p$, we obtain

$$\Pi_p K\tilde{\Sigma}_q \neq U_p. \tag{3}$$

It follows from (3) that, for some $i \in \{1, \ldots, m\}$, we have $\Pi_p^{(i)} K\tilde{\Sigma}_q \neq U_p^{(i)}$. Consider the most complicated case, in which $\dim U_p^{(i)} = 2$ and $\dim \Pi_p^{(i)} K\tilde{\Sigma}_q = 1$. Let $e_p \in U_p^{(i)}$ denote the unit vector perpendicular to $\Pi_p^{(i)} K\tilde{\Sigma}_q$, and let $\Pi_p^{e_p}$ denote the projector onto the straight line passing through the vector e_p parallel to $\Pi_p^{(i)} K\tilde{\Sigma}_q$. By the choice of e_p, we have

$$\Pi_p^{e_p} K\tilde{\Sigma}_q = \{0\}. \tag{4}$$

Any vector $x \in \Sigma_q$ can be represented as $x = \Pi_q x + y$, where $y \in \tilde{\Sigma}_q$. It follows that $\Pi_p^{e_p} Kx = \Pi_p^{e_p} K(\Pi_q x + y) = \Pi_p^{e_p} Ky$. Equality (4) implies

$$\Pi_p^{e_p} Kx = \Pi_p^{e_p} K\Pi_q x \quad \text{for} \quad x \in \Sigma_q. \tag{5}$$

Since $\Sigma_p = K\Sigma_q = K(\tilde{\Sigma}_q + S_q)$, we have

$$\Pi_p^{e_p} KS_q \neq \{0\}. \tag{6}$$

In what follows, we refer only to relations (5) and (6); the other cases differ only in the choice of the vector e_p.

We identify the straight line passing through e_p with the real line and assume that $\Pi_p^{e_p} e_p = 1$. Choose a unit vector $e_q \in S_q$ so that, for all $j \in \{1, \ldots, l\}$,

(i) $\Pi_q^{(j)} = 0$ if $\Pi_p^{e_p} KS_q^{(j)} = \{0\}$;

(ii) $\Pi_p^{e_p} K\Pi_q^{(j)} e_q < 0$ if $\Pi_p^{e_p} KS_q^{(j)} \neq \{0\}$; moreover, if $\dim S_q^{(j)} = 2$, then we choose e_q so that $\Pi_q^{(j)} e_q \perp \operatorname{Ker} \Pi_p^{e_p} K\Pi_q^{(j)}$.

Relation (6) implies the existence of $e_q \neq 0$. For each $d > 0$, consider the pseudo-orbit $g(t)$ defined by

$$g(t) = \begin{cases} \phi(t, a_q + de_q) & \text{if} \quad t < 0, \\ \phi(t, a_q) & \text{if} \quad 0 \le t < \tau, \\ \phi(t, a_p + de_p) & \text{if} \quad t \ge \tau. \end{cases}$$

Clearly, there exists a constant $C_1 \ge 1$ depending only on the flow ϕ and not depending on the choice of d, e_p, and e_q such that $g(t)$ is a $C_1 d$-pseudo-orbit of the flow ϕ.

Suppose that the field X has the Lipschitz shadowing property with constants L_0 and D_0. Suppose also that, in accordance with the assumption made above, the pseudo-orbit $g(t)$ is shadowed by the orbit of the point w_q with a reparameterization $h(t) \in \operatorname{Rep}(L_0 C_1 d)$ for $D_0/C_1 > d > 0$. We have

$$\operatorname{dist}(\phi(h(t), w_q), g(t)) \le L_0 C_1 d, \quad t \in \mathbf{R}. \tag{7}$$

Clearly, the orbit of ω_q intersects Σ_q. We denote the intersection point by ω_q'. Inequality (7) implies the existence of a constant C_2 not depending on d such that $\omega_q' = \phi(H, \omega_q)$ for some $|H| < C_2 d$. The orbit of the point ω_q' shadows the pseudo-orbit $g(t)$ with a reparameterization of class $\operatorname{Rep}(L'C_1 d)$, where $L' = (L_0C_1 + C_2)/C_1$. For simplicity, we denote ω_q' by ω_q and L' by L_0. Consider $w_p \in \Sigma_p$ defined by $w_p = Kw_q = \phi(\tau, w_q)$. The inclusion $g(\tau) \in \Sigma_p$ and inequality (7) with $t = \tau$ imply $\operatorname{dist}(\phi(h(\tau), w_q), \Sigma_p) \le L_0C_1d$. Clearly, in this case, there exists a constant C_3 not depending on d such that $w_p = \phi(h(\tau) + H, w_q)$ for some $|H| < C_3 d$.

Let $\phi_q^{(j)}(t, x) = \Pi_q^{(j)}\phi(t, \Pi_q^{(j)}x)$ be the projection of the flow ϕ on the subspace $S_q^{(j)}$. Clearly, $\phi_q^{(j)}$ is determined by a linear vector field, until the orbit leaves the neighborhood N_q. Similarly, we set $\phi_p(t, x) = \Pi_p^{(i)}\phi(t, \Pi_p^{(i)}x)$.

Take $\varepsilon = \pi/4$ and $L = C_1L_0 + 1$. We apply Lemmas 4 and 5 to these numbers and the flows $\phi_p(t, x)$ and $\phi_q^{(j)}(-t, x)$ for $j \in \{1, \dots, l\}$. Let $T = T(\varepsilon, L)$ and $d_0 = d_0(\varepsilon, L)$ be numbers such that the assertions of Lemmas 4 and 5 with these T and d_0 hold for all systems under consideration.

Choose $d_1 \in \mathbf{R}$ so that $d_0 > d_1 > 0$ and, for any $d \le d_1$, inequality (7) holds and, moreover,

$$B(L_0C_1d, \phi(t, a_q + de_q)) \subset N_q \quad \text{for} \quad 0 \ge t \ge -2T$$

and

$$B(L_0C_1d, \phi(t, a_p + de_p)) \subset N_p \quad \text{for} \quad 0 \le t \le 2T,$$

where $B(a, x)$ is the ball of radius a centered at x. This, together with (7), implies the inclusions

$$\phi(h(-t), \omega_q) \in N_q \quad \text{and} \quad \phi(h(\tau + t) - h(\tau), \omega_p) \in N_p$$

for $0 \le t \le T$. Thus, the pieces of the orbit and the pseudo-orbit of interest to us are contained in N_p and N_q.

Inequalities (7) and the definition of $g(t)$ imply

$$\left|\phi_q^{(j)}(h(t), w_q) - \phi_q^{(j)}(t, a_q + de_q)\right| \le L_0C_1d \quad \text{for} \quad -T \le t \le 0, \quad j \in \{1, \dots, l\}.$$

Let us show that $\Pi_p^{e_p}K\Pi_q^{(j)}de_q$ and $\Pi_p^{e_p}K\Omega_q^{(j)}\omega_q$ are of the same sign. Consider the more complicated case of $\dim S_q^{(j)} = 2$. Let us apply Lemma 4 to the flow $\phi_q^{(j)}(-t, x)$ with $x_0^{(j)} = \Pi_q^{(j)}(de_q)$ and $x_1^{(j)} = \Pi_q^{(j)}\omega_q$. We see that $|\arg(x_1^{(j)}) - \arg(x_0^{(j)})| < \varepsilon = \pi/4$. It follows from the choice of e_q that e_q and ω_q belong to the same half-plane with respect to $\operatorname{Ker}\Pi_p^{e_p}K\Pi_q^{(j)}$. This implies that $\Pi_p^{e_p}K\Pi_q^{(j)}e_q$ and $\Pi_p^{e_p}K\Pi_q^{(j)}\omega_q$ are of the same sign, i.e.,

$$\Pi_p^{e_p}K\Pi_q^{(j)}\omega_q < 0. \tag{8}$$

A similar argument proves that $\Pi_p^{e_p}\omega_p$ and $\Pi_p^{e_p}de_p$ are of the same sign, i.e., $\Pi_p^{e_p}\omega_p > 0$. Summing inequalities (8) over all $j \in \{1, \dots, l\}$, we obtain $\Pi_p^{e_p}K\Pi_q\omega_q < 0$. It follows from (5) that $\Pi_p^{e_p}K\omega_q < 0$; however, $\Pi_p^{e_p}K\omega_q = \Pi_p^{e_p}\omega_p > 0$. This contradiction proves Lemma 2 and Theorem 1.

4. PROOF OF THEOREM 2

To prove Theorem 2, we need two additional lemmas.

Lemma 6. Suppose that p and q are hyperbolic rest points of the vector field X and p is not a sink. Let $r = W^u(q) \cap W^s(p)$. Suppose that, in some neighborhood V of the point r,

$$W^u(q) \cap V \subset W^s(p) \cap V. \tag{9}$$

Then, $X \notin \operatorname{Int}^1(\mathrm{OrSh})$.

Proof. Without loss of generality, we can assume that $r \in W^s_{\text{loc}}(p)$ (where $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(p)$ are, respectively, the locally stable and the locally unstable manifold of the point p). Consider any point $\alpha \in W^u_{\text{loc}}(p)$. Choose $\varepsilon > 0$ so that

(i) $\operatorname{dist}(a, W^s_{\text{loc}}(p)) > \varepsilon$ and $B(\varepsilon, r) \subset V$;

(ii) the orbit of any point $x \notin W^u(q)$ as time tends to $-\infty$ leaves the ε-neighborhood of q.

For any $\tau_0, \tau_1 > 0$ consider the pseudo-orbit $g(t)$ defined by

$$g(t) = \begin{cases} \phi(t, r) & \text{if} \quad t \le \tau_0, \\ \phi(t - \tau_0 - \tau_1, \alpha) & \text{if} \quad t > \tau_0. \end{cases}$$

Since $\phi(t, r) \longrightarrow p$ as $t \longrightarrow \infty$ and $\phi(t, \alpha) \longrightarrow p$ as $t \longrightarrow -\infty$, it follows that, for any $d > 0$, there exist τ_0 and τ_1 for which $g(t)$ is a d-pseudo-orbit. Let us show that, for any reparameterization $h(t)$ and any point $x \in M$, there exists a $t \in \mathbf{R}$ for which $\operatorname{dist}(g(t), \phi(h(t), x)) > \varepsilon$. Suppose that, on the contrary,

$$\operatorname{dist}(g(t), \phi(h(t), x)) \le \varepsilon, \quad t \in \mathbf{R}. \tag{10}$$

Since $g(t) \longrightarrow q$ as $t \longrightarrow -\infty$, it follows from inequality (10) that $x \in W^u(q)$. Substituting $t = 0$ into (10), we obtain $\operatorname{dist}(r, \phi(h(0), x)) \le \varepsilon$. This inequality and relation (9) imply $\phi(h(0), x) \in W^s_{\text{loc}}(p)$; hence, $\phi(h(t), x) \in W^s_{\text{loc}}(p)$ for any $t > 0$. Therefore, by the choice of α, inequality (10) does not hold for $t = \tau_0 + \tau_1$. This implies $X \notin \operatorname{Int}^1(\text{OrSh})$.

Lemma 7. Suppose that p and q are hyperbolic rest points of a vector field $X \in \operatorname{Int}^1(\text{OrSh})$ and $\dim W^u(p) = 1$. Let $r \in W^u(q) \cap W^s(p)$. Then, r is a transversal intersection point of $W^u(q)$ and $W^s(p)$.

Proof. Suppose that r is a nontransversal intersection point of $W^u(q)$ and $W^s(p)$. As in the proof of Lemma 2, we can assume that

(i) the field X is linear in some neighborhood U of the point p and, moreover, $r \in U$;

(ii) in some neighborhood V of the point r, the manifold $W^u(q)$ has the form $r + K$, where K is a linear subspace.

Since $W^u(q)$ and $W^s(p)$ are affine spaces in the neighborhood V with $\dim W^s(p) = \dim M - 1$, it follows from the nontransversality of the intersection of $W^u(q)$ and $W^s(p)$ that $W^u(q) \cap V \subset W^s(p) \cap V$. This relation and Lemma 6 imply $X \notin \operatorname{Int}^1(\text{OrSh})$.

Proof of Theorem 2. Consider a manifold M of dimension $\dim M \le 3$. As in the proof of Theorem 1, it suffices to prove that if $X \in \operatorname{Int}^1(\text{OrSh})$, $p, q \in \operatorname{Per}(X)$, and $r \in W^u(q) \cap W^s(p)$, then r is a transversal intersection point of $W^s(p)$ and $W^u(q)$. If p or q is a closed orbit, then the required assertion follows from Lemma 3. Thus, we can assume that p and q are rest points. Suppose that $\dim M = 3$ (in the cases $\dim M = 2$ and $\dim M = 1$, the proof is similar). The following cases are possible.

(i) At least one of the manifolds $W^s(p)$ and $W^u(q)$ has dimension 3. Then their intersection is transversal.

(ii) At least one of the manifolds $W^s(p)$ and $W^u(q)$ has dimension 2. Without loss of generality, we can assume that $\dim W^s(p) = 2$. Then, by Lemma 7, the intersection of $W^s(p)$ and $W^u(q)$ is transversal.

(iii) Both manifolds $W^s(p)$ and $W^u(q)$ have dimension 1. In this case, each of these manifolds is the orbit of some point, and therefore $W^s(p) = W^u(q)$. Lemma 6 implies $X \notin \operatorname{Int}^1(\text{OrSh})$. This completes the proof of Theorem 2.

5. CONCLUSION

In this paper, the structure of the C_1-interiors of sets of vector fields with the Lipschitz and oriented shadowing properties is described.

REFERENCES

1. D. V. Anosov, in *Proceedings of the 5th International Conference on Nonlinear Oscillations, Kiev, Ukraine, 1970* (Kiev, 1970), Vol. 2, pp. 39–45.

2. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (Springer, Berlin, 1975).
3. S. Yu. Pilyugin, *Shadowing in Dynamical Systems* (Springer, Berlin, 1999).
4. K. Palmer, *Shadowing in Dynamical Systems: Theory and Applications* (Kluwer, Dordrecht, 2000).
5. S. Yu. Pilyugin, *An Introduction to Structurally Stable Systems of Differential Equations* (Leningrad. Univ., Leningrad, 1988).
6. S. Yu. Pilyugin, J. Differ. Equat. **140**, 238–265 (1997).
7. S. Yu. Pilyugin, A. A. Rodionova, and K. Sakai, Discr. Contin. Dyn. Systems **9**, 287–308 (2003).
8. S. Gan, Sci. China Ser A **41**, 1076–1082 (1998).
9. K. Lee and K. Sakai, J. Differ. Equat. **232**, 303–313 (2007).