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Systems of Reproducing Kernels and their Biorthogonal: Completeness or Incompleteness?

Anton Baranov[1] and Yurii Belov[2]

[1]Department of Mathematics and Mechanics, Saint Petersburg State Univerisity, 28, Universitetski pr., St. Petersburg 198504, Russia and [2]Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), NO-7491 Trondheim, Norway

Correspondence to be sent to: e-mail: anton.d.baranov@gmail.com

Let $\{v_n\}$ be a complete minimal system in a Hilbert space $\mathcal{H}$ and let $\{w_m\}$ be its biorthogonal system. It is well known that $\{w_m\}$ is not necessarily complete. However, the situation may change if we consider systems of reproducing kernels in a reproducing kernel Hilbert space $\mathcal{H}$ of analytic functions. We study the completeness problem for a class of spaces with a Riesz basis of reproducing kernels and for model subspaces K_Θ of the Hardy space. We find a class of spaces where systems biorthogonal to complete systems of reproducing kernels are always complete, and show that in general this is not true. In particular, we answer the question posed by Nikolski and construct a model subspace with an incomplete biorthogonal system.

1 Introduction and Main Results

1.1 Statement of the problem

Let $\mathcal{H}$ be a separable Hilbert space. A sequence of vectors $\{v_n\}$ is said to be *complete* if $\overline{\operatorname{Span}\{v_n\}} = \mathcal{H}$. If, moreover, the system $\{v_n\}$ is *minimal*, that is, it fails to be complete when we remove any vector, then we say that the system is *exact*. For every

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exact system of vectors $\{v_n\}$, there exists a unique biorthogonal system $\{w_m\}$ such that $\langle v_n, w_m\rangle = \delta_{mn}$.

Suppose that $\mathcal{H}$ is a space of entire functions with reproducing kernels. Namely, for each $\lambda \in \mathbb{C}$ there is an element $k_\lambda \in \mathcal{H}$ such that $\langle f, k_\lambda\rangle = f(\lambda)$ for all $f \in \mathcal{H}$. We are looking for an answer to the following question:

Question. Let $\{k_{\lambda_n}\}$ be an exact system of reproducing kernels in $\mathcal{H}$. Is it true that its biorthogonal system is also complete in $\mathcal{H}$? □

Of course, for an arbitrary sequence of vectors its biorthogonal system may be incomplete. If $\{e_n\}_{n=1}^\infty$ is an orthonormal basis, then the system $\{e_n + e_1\}_{n=2}^\infty$ is complete, but the biorthogonal system $\{e_n\}_{n=2}^\infty$ is incomplete. On the other hand, it is well known that if we restrict ourselves to *systems of reproducing kernels*, then the answer may be positive. Young [21] proved the completeness of systems biorthogonal to the systems of reproducing kernels in the Paley–Wiener spaces or, equivalently, for biorthogonals to systems of exponentials in L^2 on an interval. Fricain [10] extended this result to a class of de Branges spaces of entire functions (see discussion below).

Our aim is to exhibit some classes of spaces for which we know the answer (positive or negative). In particular, we answer the question posed by Nikolski and construct an example of a model (shift-coinvariant) subspace of the Hardy space H^2 with an incomplete biorthogonal system. In what follows we consider only systems biorthogonal to systems of reproducing kernels; therefore, sometimes we write simply *the biorthogonal system* in place of *the system biorthogonal to an exact system of reproducing kernels*.

To make the problem more realistic we need some additional structure on $\mathcal{H}$, namely the existence of a *Riesz basis*. Recall that a system of vectors $\{v_n\}$ is said to be a Riesz basis if $\{v_n\}$ is an image of an orthonormal basis under a bounded and invertible operator in $\mathcal{H}$. We consider the class $\mathfrak{R}$ of spaces of entire functions satisfying three axioms:

(A1) $\mathcal{H}$ has a reproducing kernel k_λ at every point $\lambda \in \mathbb{C}$.

(A2) If a function f is in $\mathcal{H}$ and $f(w) = 0$, then the function $\frac{f(z)}{z-w}$ is also in $\mathcal{H}$.

(A3) There exists a sequence of distinct points $T = \{t_n\} \subset \mathbb{C}$ such that the sequence of normalized reproducing kernels $\{k_{t_n}/\|k_{t_n}\|_{\mathcal{H}}\}$ is a Riesz basis for $\mathcal{H}$.

First example of such spaces is the Paley–Wiener space PW_π which is the space of entire functions of exponential type at most π that are in $L^2(\mathbb{R})$. In this case, the sequence $\{\frac{\sin(\pi(z-n))}{\pi(z-n)}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of reproducing kernels and (A3) is satisfied. Axioms (A1) and (A2) follow immediately.

More general examples are de Branges spaces. We say that an entire function E belongs to the Hermite–Biehler class if it has no real zeros and $|E(z)| > |E(\bar z)|$ for any z in the upper half-plane $\mathbb{C}^+$. The de Branges space $\mathcal{H}(E)$ consists of all entire functions f such that f/E and f^*/E belong to the Hardy space H^2 in $\mathbb{C}^+$; here, $f^*(z) = \overline{f(\bar z)}$. The norm in $\mathcal{H}(E)$ is given by

$$\|f\|^2_{\mathcal{H}(E)} = \int_{\mathbb{R}} \frac{|f(x)|^2}{|E(x)|^2}\, \mathrm{d}x.$$

As in the Paley–Wiener space, in de Branges spaces there exist orthonormal bases of reproducing kernels (see [8]). So de Branges spaces form a subclass of our class $\mathfrak{R}$.

1.2 Parametrization of the class $\mathfrak{R}$

We will use an explicit parametrization of the class $\mathfrak{R}$ from [6]. Let us briefly remind the description from [6].

The Riesz basis $\{k_{t_n}/\|k_{t_n}\|_{\mathcal{H}}\}$ has a biorthogonal basis, which we will call $\{f_n\}$. Using axiom (A2) we conclude that $f_n(z)\frac{(z-t_n)}{(z-t_m)} \in \mathcal{H}$, $m \neq n$. This function vanishes at the points t_l, $l \neq m$, and so it equals f_m up to a multiplicative constant. Hence, the function $c_m f_m(z)(z-t_m)$ does not depend on m for suitable coefficients c_m. We call this function the *generating function* of the sequence $\{t_n\}$ and denote it by F. Note that using this construction we may define the generating function for an arbitrary exact system of reproducing kernels, not just for a Riesz basis.

The sequence f_n is also a Riesz basis for $\mathcal{H}$, and therefore any vector h in $\mathcal{H}$ can be written as

$$h(z) = \sum_n h(t_n) \frac{F(z)}{F'(t_n)(z-t_n)}, \tag{1.1}$$

where the sum converges with respect to the norm of $\mathcal{H}$ and

$$A \sum_n \frac{|h(t_n)|^2}{\|k_{t_n}\|^2_{\mathcal{H}}} \le \|h\|^2_{\mathcal{H}} \le B \sum_n \frac{|h(t_n)|^2}{\|k_{t_n}\|^2_{\mathcal{H}}}$$

for some constants $A, B > 0$ independent of h. Since point evaluation at every point z is a bounded linear functional, the series in (1.1) also converges uniformly on compact subsets of $\mathbb{C} \setminus T$. By the assumption that $h \mapsto \{h(t_n)/\|k_{t_n}\|_{\mathcal{H}}\}$ is a bijective map from $\mathcal{H}$ to ℓ^2, we get

$$\sum_n \frac{\|k_{t_n}\|^2_{\mathcal{H}}}{|F'(t_n)|^2|z-t_n|^2} < +\infty \tag{1.2}$$

whenever z is in $\mathbb{C} \setminus T$. Therefore (1.2) implies that

$$\sum_n \frac{b_n}{1+|t_n|^2} < +\infty, \quad b_n := \frac{\|k_{t_n}\|^2_{\mathcal{H}}}{|F'(t_n)|^2}. \tag{1.3}$$

It follows from (1.1) that we can associate with the space $\mathcal{H} \in \mathfrak{R}$ a space of *meromorphic functions with prescribed poles*. Namely, given a sequence of distinct complex numbers $T = \{t_n\}$ and a weight sequence $b = \{b_n\}$ which satisfy the admissibility condition (1.3), we introduce the space $\mathcal{H}(T, b)$ consisting of all functions of the form

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n b_n^{1/2}}{z - t_n} \tag{1.4}$$

such that

$$\|f\|^2_{\mathcal{H}(T,b)} := \sum_{n=1}^{\infty} |a_n|^2 < +\infty.$$

The map $f \mapsto Ff$ is an isomorphism of $\mathcal{H}(T, b)$ onto $\mathcal{H}$ which maps reproducing kernels to reproducing kernels. So, for our approach, we can consider the pairs (T, b) as a parametrization of all spaces from $\mathfrak{R}$.

The space $\mathcal{H} \in \mathfrak{R}$ is a de Branges space $\mathcal{H}(E)$ for some Hermite–Biehler function E if and only if there exists T such that $T \subset \mathbb{R}$. In this case, we may choose E so that $F = \frac{E+E^*}{2}$. Moreover, as was shown in [5], de Branges spaces are the only spaces of the class $\mathfrak{R}$ where there exist two different orthogonal bases of reproducing kernels. These spaces are the prime example for us. Note, however, that there are many spaces in the class $\mathfrak{R}$ which are not isomorphic to a de Branges space. For example, let $T = \{u_n\} \cup \{iw_n\}$, where u_n and w_n are arbitrary sequences of real points satisfying $\sum_n |u_n|^{-1} = \sum_n |w_n|^{-1} = \infty$. Then the space $F\mathcal{H}(T, b)$ is not a de Branges space (in any half-plane) since for a Riesz sequence of normalized kernels $\{k_{\lambda_n}/\|k_{\lambda_n}\|_{\mathcal{H}}\}$ in a de Branges space $\mathcal{H}$, the sequences $\{\lambda_n\} \cap \mathbb{C}^+$ and $\{\lambda_n\} \cap \mathbb{C}^-$ should satisfy the Carleson interpolation condition [19, Part D, Lemma 4.4.2].

1.3 Main theorems

Now we are ready to state our main results.

Theorem 1.1. If $\mathcal{H} \in \mathfrak{R}$ and $\sum_n b_n < +\infty$, then there exists an exact system of reproducing kernels such that its biorthogonal system is not complete. □

A converse result says that if b_n have no more than a power decay, then the biorthogonal systems are (almost) complete.

Theorem 1.2. If there exists $N > 0$ such that

$$\inf_m b_m (1 + |t_m|)^N > 0,$$

then the orthogonal complement to a system biorthogonal to an exact system of reproducing kernels is finite-dimensional.

If, moreover, $\sum_n b_n = +\infty$, then any system biorthogonal to an exact system of reproducing kernels is complete in $\mathcal{H}$. □

The restriction on the decay of b_n in Theorem 1.2 is essential.

Example 1.3. There exists a space $\mathcal{H} \in \mathfrak{R}$ such that $\sum_n b_n = +\infty$, but there exists an exact system of reproducing kernels such that its biorthogonal is not complete. □

Young's result about the Paley–Wiener space corresponds to the situation when $t_n = n$, $n \in \mathbb{Z}$, and $b_n = 1$, and follows from Theorem 1.2. Fricain have proved the completeness of biorthogonal system in de Branges spaces under the assumption that $\sup_{x\in\mathbb{R}} \varphi'(x) < +\infty$, φ being the phase function for E, that is, a smooth branch of the argument of E on $\mathbb{R}$. We show now that this assumption implies a lower estimate on b_n. We will use the de Branges decomposition $E = A - \mathrm{i}B$, where A and B are entire functions real on the real axis. As known, reproducing kernels in the de Branges space $\mathcal{H}(E)$ are given by

$$k_w(z) = \frac{1}{\pi} \cdot \frac{B(z) \cdot A(\bar{w}) - A(z)B(\bar{w})}{z - \bar{w}}. \tag{1.5}$$

Without loss of generality, we may assume that $\{k_{t_n}/\|k_{t_n}\|\}$ is an orthonormal basis, where $t_n \in \mathbb{R}$ are all solutions of the equation $A(z) = 0$ [8, Theorem 22]. Of course, A is the corresponding generating function. In this notation

$$b_n = \frac{\|k_{t_n}\|^2}{|A'(t_n)|^2} = \frac{k_{t_n}(t_n)}{|A'(t_n)|^2} = \frac{B(t_n)}{\pi A'(t_n)} = \frac{1}{\pi \varphi'(t_n)}. \tag{1.6}$$

Hence, $\inf_n b_n > 0$ and the result follows from Theorem 1.2.

1.4 Size of the orthogonal complement

Now we turn to the question of "size" of the orthogonal complement to a biorthogonal system in the case when it is not complete. A precise definition of the "size" and the main results are given in §3. Here, we only emphasize the following informal principle:

> The size of the orthogonal complement to a biorthogonal system depends on smallness of the sequence $\{b_n\}$. The orthogonal complement becomes bigger if b_n tend to zero faster. If, however, $\{b_n\}$ are extremely small, then the orthogonal complement is finite-dimensional.

Now we give a precise formulation of the latter property. Put

$$\ell^2(T, b) := \left\{ f : T \to \mathbb{C} : \sum_n |f(t_n)|^2 b_n < +\infty \right\}.$$

The following result relates the size of the biorthogonal system to the density of polynomials on discrete subsets of the real line. Assume that b_n are so small that the polynomials belong to $\ell^2(T, b)$ and are dense there, that is, there is no nontrivial sequence $\{c_n\} \in \ell^2(T, b)$, such that $\sum_n c_n n^k = 0$ for any $k \in \mathbb{N}_0$.

Theorem 1.4. Let $T \subset \mathbb{R}$ and assume that the polynomials are dense in $\ell^2(T, b)$. If $\mathcal{H}$ is the Hilbert space of the class $\mathfrak{R}$ corresponding to $\mathcal{H}(T, b)$, then the closed linear span of the system biorthogonal to an exact system of reproducing kernels always has a finite codimension. □

Density of polynomials in the spaces of the form $\ell^2(T, b)$ is closely connected to the quasianalyticity phenomena (see [7, 13]). We illustrate this by the following example (further examples are given in Section 3).

Example 1.5. Let $t_n = n$, $n \in \mathbb{Z}$, and let $b_n = \exp(-|n|)$. Then any biorthogonal system has a finite codimension. More generally, let $w = \exp(-\Omega)$ be an even function such that $\Omega(e^t)$ is a convex function of t (w is a so-called normal majorant). Let $b_n = w(n)$. If $\sum_n \frac{|\log b_n|}{n^2+1} = +\infty$, then any biorthogonal system has a finite codimension. □

Finally, let us summarize:

(i) If b_n has at most power decay, then any biorthogonal system has a finite codimension (Theorem 1.2).

(ii) If the polynomials belong to $\ell^2(T, b)$ and are not dense there ("nonquasianalytic case"), then the codimension may be infinite (see Proposition 3.4).

(iii) If the polynomials are dense in $\ell^2(T, b)$ ("quasianalytic case"), then the codimension is finite (Theorem 1.4).

1.5 Model subspaces

Our next result is about general *model* or *star-invariant* subspaces K_Θ of the Hardy space H^2 in the upper half-plane $\mathbb{C}^+$. Let Θ be an inner function in $\mathbb{C}^+$, that is, a bounded analytic function such that $\lim_{y\to+0} |\Theta(x+iy)| = 1$ for almost all $x \in \mathbb{R}$. With each Θ we associate the subspace

$$K_\Theta = H^2 \ominus \Theta H^2.$$

These subspaces, as well as their vector-valued generalizations play an outstanding role both in function theory and operator theory. For their numerous applications, we refer to [17–19]. It is well known that if Θ has a meromorphic continuation to the whole plane, then $\Theta = E^*/E$ for a function E in the Hermite–Biehler class and the mapping $f \mapsto Ef$ is a unitary operator from K_Θ onto $\mathcal{H}(E)$, which maps reproducing kernels onto reproducing kernels.

The reproducing kernels of the space K_Θ are of the form

$$k_\lambda(z) = \frac{i}{2\pi} \cdot \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{z - \bar{\lambda}}, \qquad \lambda \in \mathbb{C}^+. \tag{1.7}$$

Reproducing kernels of the model spaces have a rich and subtle structure and their geometric properties (such as completeness, Bessel sequences, Riesz basic sequences) are still not completely understood (see, e.g., [2, 12, 15] and [19, Part D, Chapter 4]). In particular, it is an open problem whether any model subspace has a Riesz basis of reproducing kernels. A special case of the completeness problem for reproducing kernels is the completeness of exponential systems in $L^2(-a, a)$ (corresponding to $\Theta(z) = e^{2iaz}$) settled by the classical Beurling–Malliavin theory. A recent breakthrough in the completeness problem for reproducing kernels in model subspaces is due to Makarov and Poltoratski [15, 16], who suggested a new approach based on singular integrals and extended the Beurling–Malliavin theory to some classes of model subspaces and de Branges spaces.

The problem whether the system biorthogonal to an exact system of reproducing kernels is complete in K_Θ was posed by Nikolski; it was studied by Fricain in [10] where for the class of inner functions with bounded derivatives a positive answer was obtained. Theorem 1.1, which applies to the case of meromorphic inner functions, already shows that the answer in general is negative. However, we are able to prove an analog of Theorem 1.1 for general model spaces.

Let $\sigma(\Theta)$ be the spectrum of the inner function Θ, that is, the set of all $\zeta \in \mathbb{R} \cup \infty$ such that $\lim_{z\to\zeta} \inf |\Theta(z)| = 0$. Note that $\sigma(\Theta)$ is closed and Θ (and any $f \in K_\Theta$) has analytic continuation across any interval of the set $\mathbb{R} \setminus \sigma(\Theta)$. A point $\zeta \in \mathbb{R}$ is said to be a *Carathéodory point* for Θ if Θ has an angular derivative at ζ, that is, there exists the nontangential limit $\Theta(\zeta)$ with $|\Theta(\zeta)| = 1$, as well as the nontangential limit $\Theta'(\zeta) = \lim_{z\to\zeta} \frac{\Theta(z)-\Theta(\zeta)}{z-\zeta}$.

Theorem 1.6. Let Θ be an inner function in $\mathbb{C}^+$ such that there exists $\zeta \in \sigma(\Theta)$ which is a Carathéodory point for Θ. Then there exists an exact system of reproducing kernels such that the biorthogonal system is not complete. □

If $\sigma(\Theta) = \{\infty\}$ (i.e., Θ is meromorphic in $\mathbb{C}$) the existence of the "angular derivative at ∞" (appropriately defined) is equivalent to the condition $\sum_n b_n < +\infty$ for some orthogonal basis of reproducing kernels and we arrive at Theorem 1.1 for de Branges spaces. We mention also that a result analogous to Theorem 1.6 holds for the model spaces in the unit disc (see Theorem 4.1).

In Section 5, Theorem 5.3, we obtain a condition sufficient for the completeness of a biorthogonal system in a general model space K_Θ in terms of the generating function G.

Throughout this paper, the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant $C > 0$ such that $U(z) \le C V(z)$ holds for all suitable z. We write $U(z) \asymp V(z)$ if $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2 Proof of Theorems 1.1 and 1.2

Let $\{k_{\lambda_n}\}$ be an exact system in $\mathcal{H}$. Without loss of generality, we can assume that $\{\lambda_n\}\cap\{t_m\}=\emptyset$, since we always can move slightly the points $\{t_m\}$ so that $\{k_{t_m}/\|k_{t_m}\|_{\mathcal{H}}\}$ remains a Riesz basis. Suppose that F and G are generating functions of systems $\{k_{t_m}\}$ and $\{k_{\lambda_n}\}$, respectively. Then the system $\frac{G(z)}{G'(\lambda_n)(z-\lambda_n)}$ is biorthogonal to $\{k_{\lambda_n}\}$.

For any $h\in\mathcal{H}$ we have a representation with respect to the Riesz basis $\{k_{t_m}/\|k_{t_m}\|_{\mathcal{H}}\}$:

$$h=\sum_m \bar{a}_m\frac{k_{t_m}}{\|k_{t_m}\|_{\mathcal{H}}},\qquad \{a_m\}\in\ell^2. \tag{2.1}$$

The last series converges in the norm and pointwise.

The function h is orthogonal to $\frac{G(z)}{z-\lambda_n}$ for all n if and only if for any n

$$\left\langle\frac{G}{z-\lambda_n},h\right\rangle=\sum_m\frac{a_m}{\|k_{t_m}\|}\cdot\frac{G(t_m)}{t_m-\lambda_n}=0.$$

Consider the meromorphic function

$$L(z):=\sum_m\frac{a_m}{\|k_{t_m}\|_{\mathcal{H}}}\cdot\frac{G(t_m)}{z-t_m}.$$

The series converges uniformly on compact subsets of $\mathbb{C}\setminus T$, since $\frac{G}{z-\lambda_n}\in\mathcal{H}$ and so $\{\frac{G(t_m)}{t_m\|k_{t_m}\|_{\mathcal{H}}}\}\in\ell^2$. The function LF is entire and vanishes at the points $\{\lambda_n\}$. Hence, $S:=LF/G$ is an entire function, and

$$\frac{G(z)S(z)}{F(z)}=\sum_m\frac{a_m}{\|k_{t_m}\|_{\mathcal{H}}}\cdot\frac{G(t_m)}{z-t_m}. \tag{2.2}$$

It follows that $a_m=S(t_m)\frac{\|k_{t_m}\|_{\mathcal{H}}}{F'(t_m)}=\frac{|F'(t_m)|}{F'(t_m)}S(t_m)b_m^{1/2}$. Hence,

$$\sum_m|S(t_m)|^2b_m<+\infty. \tag{2.3}$$

We can consider functions S from (2.2) which satisfy (2.3) as parametrization of all functions h orthogonal to a given biorthogonal system $\{\frac{G(z)}{z-\lambda_n}\}$. We denote the space of all such functions S by $\mathcal{S}$. It is a Hilbert space with respect to the norm given as the square root of the left-hand side of (2.3). Moreover, the mapping

$$S \mapsto \sum_m \frac{|F'(t_m)|}{F'(t_m)} \cdot \overline{S(t_m)}\, b_m^{1/2} \cdot \frac{k_{t_m}}{\|k_{t_m}\|_{\mathcal{H}}}$$

is a unitary operator from $\mathcal{S}$ onto the orthogonal complement of the system $\{\frac{G(z)}{z-\lambda_n}\}$.

Now we turn to the proof of Theorem 1.1. From (1.4) we get the following.

Proposition 2.1. Function M is in $\mathcal{H}$ if and only if

$$\frac{M(z)}{F(z)} = \sum_n \frac{c_n}{z-t_n}, \qquad \sum_n \frac{|c_n|^2}{b_n} < +\infty. \tag{2.4}$$

Here the series converges uniformly on compact sets in $\mathbb{C} \setminus T$, while the series $\sum_n c_n \frac{F(z)}{z-t_n}$ converges in the norm of the space $\mathcal{H}$. □

Proof of Theorem 1.1. We want to construct a generating function G of an exact system of reproducing kernels such that $\frac{G(z)}{F(z)} = \sum_n \frac{d_n}{z-t_n}$, where the series converges uniformly on compact subsets of $\mathbb{C} \setminus T$. If such a function G is constructed, we can take $S \equiv 1$ in (2.2) and the function

$$h = \sum_n b_n^{1/2} \cdot \frac{|F'(t_n)|}{F'(t_n)} \cdot \frac{k_{t_n}}{\|k_{t_n}\|_{\mathcal{H}}}$$

is orthogonal to all $\frac{G(z)}{z-\lambda_n}$.

We choose coefficients c_n so that $c_n t_n > 0$,

$$(1)\ \sum_n \frac{|c_n|^2}{b_n} < +\infty; \quad (2)\ \sum_n \frac{(c_n t_n)^2}{b_n} = +\infty, \tag{2.5}$$

and put $G(z) = F(z) \sum_n \frac{c_n t_n}{z-t_n}$. It follows from (2.5) that $G \notin \mathcal{H}$, but $\frac{G(z)}{z-\lambda_n} \in \mathcal{H}$ where λ_n are zeros of G. Without loss of generality, we can assume that G has no multiple roots because it is enough to change one coefficient c_0 a little. Indeed, we can write

$$G(z) = c_0 \frac{F(z)}{z-t_0} + H(z) = c_0 F_1(z) + H(z),$$

and the functions F_1 and H have no common zeros. So, G and G' have common zeros only at the points z where $F_1'(z)H(z) - H'(z)F_1(z) = 0$ and $c_0 F_1(z) + H(z) = 0$, but this is possible only for a countable set of coefficients c_0.

To prove the completeness of $\{k_{\lambda_n}\}$ we need to show that there is no entire function T such that $TG \in \mathcal{H}$. To prove this, we introduce some additional requirements on c_n. Choose a subsequence of indices $\{n_k\}$ so rare that $|t_{n_{k+1}}| > 2|t_{n_k}|$ and the disks $D_k^1 = \{z : |z - t_{n_k}| \le \frac{t_{n_k}}{10}\}$ are pairwise disjoint. For other indices, we choose a sequence of positive numbers h_n with $\sum_n h_n < 1$ such that the disks $D_n^2 = \{|z - t_n| \le h_n\}$, $n \notin \{n_k\}$ are also pairwise disjoint. Now assume that in addition to (2.5) we have

$$\sum_k c_{n_k} t_{n_k} < +\infty.$$

This will be achieved if we assume $\sum_k |t_{n_k}|^{-2} < +\infty$, $\sum_k b_{n_k}^{1/2} < +\infty$ and put $c_{n_k} := (b_{n_k})^{1/2} t_{n_k}^{-1}$. Now we make c_n for $n \notin \{n_k\}$ so small that

$$\sum_{n \notin \{n_k\}} \frac{|c_n t_n^2|}{h_n} < \frac{1}{10} \sum_k c_{n_k} t_{n_k}, \qquad \sum_{n \notin \{n_k\}} c_n t_n < \frac{1}{10} \sum_k c_{n_k} t_{n_k}.$$

Assume that $TG \in \mathcal{H}$. Since TG/F should be represented as in (2.4), we have

$$\frac{T(z)G(z)}{F(z)} = \sum_n \frac{c_n t_n T(t_n)}{z - t_n}, \qquad \sum_n \frac{|c_n t_n T(t_n)|^2}{b_n} < \infty. \tag{2.6}$$

We can estimate $G(z)/F(z)$ for $z \notin (\bigcup D_k^1) \cup (\bigcup D_n^2)$:

$$\begin{aligned} \frac{|G(z)|}{|F(z)|} &= \left| \sum_n \frac{c_n t_n}{z - t_n} \right| \ge \frac{1}{|z|} \sum_n c_n t_n - \sum_{n \notin \{n_k\}} \frac{|c_n t_n^2|}{|z(z - t_n)|} - \sum_{n \in \{n_k\}} \frac{|c_n t_n^2|}{|z(z - t_n)|} \\ &\ge \frac{1}{|z|} \sum_n c_n t_n - \sum_{n \notin \{n_k\}} \frac{|c_n t_n^2|}{|z| h_n} - \sum_{n \in \{n_k\}, |t_n| < |z|/3} \frac{c_n t_n}{2|z|} - 10 \sum_{n \in \{n_k\}, |t_n| \ge |z|/3} \frac{c_n t_n}{|z|}. \end{aligned}$$

Note that the last sum is $o(|z|^{-1})$ as $|z| \to \infty$. Hence, for sufficiently large $|z|$

$$\frac{|G(z)|}{|F(z)|} \ge \frac{1}{4|z|} \sum_k c_{n_k} t_{n_k} \gtrsim \frac{1}{|z|}, \qquad z \notin \left(\bigcup D_k^1 \right) \cup \left(\bigcup D_n^2 \right).$$

Using analogous estimates we can show that

$$\frac{|T(z)G(z)|}{|F(z)|} \lesssim 1, \qquad z \notin \left(\bigcup D_k^1 \right) \cup \left(\bigcup D_n^2 \right).$$

Hence, $|T(z)| \lesssim 1 + |z|$ for $z \notin (\bigcup D_k^1) \cup (\bigcup D_n^2)$. By the choice of t_{n_k} and h_n there exist circles $\Gamma_n = \{z : |z| = r_n\}$ with $r_n \to \infty$, such that $\Gamma_n \cap ((\bigcup D_k^1) \cup (\bigcup D_n^2)) = \emptyset$. Therefore, $T(z) = az + b$. But this contradicts (2.6) and (2) in (2.5) unless $T \equiv 0$. ■

Remark 2.2. Note that, by the choice of the coefficients $d_n = c_n t_n > 0$, all zeros of the function G constructed in the proof of Theorem 1.1 are real. □

In the proof of Theorem 1.2 we will use the following lemma.

Lemma 2.3. If $S \in \mathcal{S}$, then $\frac{S(z)-S(w)}{z-w} \in \mathcal{S}$ for any $w \in \mathbb{C}$. In particular, if $\mathcal{S}$ is of finite dimension $n+1$, then $\mathcal{S}$ coincides with the set $\mathcal{P}_n$ of all polynomials of degree at most n. □

Proof. Let λ_0 be a zero of G. Then $\frac{G(z)}{z-\lambda_0} \in \mathcal{H}$ and

$$\frac{G(z)}{(z-\lambda_0)F(z)} = \sum_m \frac{G(t_m)}{(t_m-\lambda_0)F'(t_m)(z-t_m)}. \tag{2.7}$$

We have the identity

$$\begin{aligned}\frac{G(z)(S(z)-S(w))}{F(z)(z-w)} &= \frac{1}{z-w}\left(\frac{G(z)S(z)}{F(z)} - \frac{G(w)S(w)}{F(w)}\right)\\ &\quad + \frac{(w-\lambda_0)S(w)}{z-w}\left(\frac{G(w)}{(w-\lambda_0)F(w)} - \frac{G(z)}{(z-\lambda_0)F(z)}\right) - S(w)\frac{G(z)}{(z-\lambda_0)F(z)}.\end{aligned}$$

By (2.2) and (2.7),

$$\frac{1}{z-w}\left(\frac{G(z)S(z)}{F(z)} - \frac{G(w)S(w)}{F(w)}\right) = \sum_m \frac{a_m}{(t_m-w)\|k_{t_m}\|_{\mathcal{H}}}\cdot\frac{G(t_m)}{t_m-z},$$

$$\frac{1}{z-w}\left(\frac{G(z)}{(z-\lambda_0)F(z)} - \frac{G(w)}{(w-\lambda_0)F(w)}\right) = \sum_m \frac{a_m}{(t_m-\lambda_0)(t_m-w)F'(t_m)}\cdot\frac{G(t_m)}{t_m-z}.$$

Thus, we have shown that the function $\frac{G(z)}{F(z)}\cdot\frac{S(z)-S(w)}{z-w}$ can be represented as a series in (2.2). Condition (2.3) for the function $\frac{S(z)-S(w)}{z-w}$ follows from (1.3). ■

Proof of Theorem 1.2. Assume that the system $\{\frac{G(z)}{z-\lambda_n}\}$ is not complete. Fix h as in (2.1) which is orthogonal to all functions $\frac{G(z)}{z-\lambda_n}$ and consider the corresponding space $\mathcal{S}$. If $\mathcal{S}$ is an infinite-dimensional space, then we can find a function $S \in \mathcal{S}$ with at least $N+2$ zeros

$w_1, \ldots, w_{N+1}$ different from the points $\{\lambda_n\}$ and $\{t_n\}$. By Lemma 2.3, $T(z) := \frac{S(z)}{\prod_{l=1}^{N+1}(z-w_l)} \in \mathcal{S}$. Moreover, the corresponding coefficients from (2.2) for T are equal to $\frac{a_m}{\prod_{l=1}^{N+1}(t_m-w_l)}$.

Recall that

$$f_m(z) := \frac{\|k_{t_m}\|_{\mathcal{H}}}{F'(t_m)} \cdot \frac{F(z)}{z-t_m} = b_m^{1/2} \cdot \frac{|F'(t_m)|}{\overline{F'(t_m)}} \cdot \frac{F(z)}{z-t_m}$$

is the biorthogonal system to the Riesz basis $\{k_{t_m}/\|k_{t_m}\|_{\mathcal{H}}\}$ and thus also is a Riesz basis. We have

$$G(z)T(z) = F(z)\sum_m \frac{a_m}{\prod_{l=1}^{N+1}(t_m-w_l)} \cdot \frac{G(t_m)}{\|k_{t_m}\|_{\mathcal{H}}(z-t_m)}$$
$$= \sum_m \frac{a_m}{b_m^{1/2}\prod_{l=1}^{N}(t_m-w_l)} \cdot \frac{G(t_m)}{(t_m-w_{N+1})\|k_{t_m}\|_{\mathcal{H}}} \cdot b_m^{1/2}\frac{F(z)}{z-t_m} =: \sum_m d_m f_m(z).$$

Note that $\inf_m |b_m^{1/2}\prod_{l=1}^{N}(t_m-w_l)| > 0$ and if we fix a zero λ_0 of G, then

$$\left|\frac{G(t_m)}{t_m-\lambda_0}\right| \le \left\|\frac{G(z)}{z-\lambda_0}\right\|_{\mathcal{H}} \cdot \|k_{t_m}\|_{\mathcal{H}}.$$

So the coefficients $\{d_m\}$ are in ℓ^2 and $GT \in \mathcal{H}$. However, this contradicts the completeness of the system $\{k_{\lambda_n}\}$.

Finally, assume that $\sum_n b_n = +\infty$. If $\mathcal{S}$ is a finite-dimensional space, then it follows from Lemma 2.3 that $\mathcal{S}$ is the space of polynomials $\mathcal{P}_n$ for some n, which cannot be true since $\sum_n |S(t_n)|^2 b_n = +\infty$ for any $S \in \mathcal{S}$. Thus, the biorthogonal system is complete. ∎

At the end of the section, we prove Example 1.3.

Proof. We will construct a space of the form $\mathcal{H}(\mathbb{Z}, b)$, that is, $t_n = n$, $n \in \mathbb{Z}$. The corresponding generating function is $F(z) = \sin(\pi z)$. Put

$$S(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{(2^k+1/2)^2}\right), \qquad G(z) = \frac{\cos(\pi z)}{S(z)}.$$

Then

$$\frac{G(z)}{\sin(\pi z)} = \sum_{n=-\infty}^{+\infty} \frac{S^{-1}(n)}{\pi(z-n)}. \tag{2.8}$$

Indeed, the difference between the left-hand side and the right-hand side in (2.8) should be an entire function of exponential type. Since $\lim_{|z|\to\infty} \frac{G(z)}{\sin \pi z} = 0$ along any nonhorizontal ray in $\mathbb{C}^+$ or $\mathbb{C}^-$, this difference is zero.

Put $b_n = |S(n)|^{-2}$ for $n \neq \pm 2^k$ and $b_n = 1$ otherwise. We consider the space $\mathcal{H} \in \mathfrak{R}$ corresponding to $\mathcal{H}(\mathbb{Z}, b)$. First of all, we want to show that G is the generating function of an exact system in $\mathcal{H}$. From (2.8) we conclude that $\frac{G(z)}{z-\lambda_0} \in \mathcal{H}$ for any zero λ_0 of G. We need to show that there is no nonzero entire T such that $TG \in \mathcal{H}(\mathbb{Z}, b)$. If it exists then

$$\frac{G(z)T(z)}{\sin(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{S^{-1}(n)T(n)}{\pi(z-n)}, \qquad \sum_n \frac{|S^{-1}(n)T(n)|^2}{b_n} < +\infty. \tag{2.9}$$

It follows from (2.9) that T is of zero exponential type, and, at the same time, $\sum_k |T(2k+1)|^2 < +\infty$, which implies $T \equiv 0$ (see, e.g., [14, Lecture 21]).

By construction, we automatically get $\sum_n b_n = +\infty$ and now it remains to show that there exists a nonzero $S_1 \in \mathcal{S}$. Let

$$S_1(z) = \prod_{k=2}^{\infty} \left(1 - \frac{z^2}{(2^k + \delta_k)^2}\right)$$

and choose $\delta_k \in (0, 1)$ so small that $\sum_{k=1}^{\infty} |S_1(2^k)|^2 < +\infty$. By a straightforward estimation we get $|S_1(n)| \lesssim |S(n)|$. So, $\sum_n |S_1(n)|^2 b_n < +\infty$. On the other hand, $\lim_{y\to\pm\infty} \frac{|S_1(iy)|}{|S(iy)|} = 0$ and we have representation (2.2) with condition (2.3) for S_1.

Since S_1 is not a polynomial, we see that $\mathcal{S}$ is infinitely dimensional. Thus, in this case the orthogonal complement to the biorthogonal system is infinitely dimensional. ■

3 Size of the Orthogonal Complement to a Biorthogonal System

In this section, we will study in more detail the space $\mathcal{S}$ the elements of which parameterize functions orthogonal to the biorthogonal system via formula (2.2). First of all note that for any $k \in \mathbb{N}_0$ it is possible that $\mathcal{S}$ coincides with the set $\mathcal{P}_k$ of polynomials of degree at most k. On the other hand, as we have seen in the proof of Example 1.3, it is possible that $\mathcal{S}$ is an infinite-dimensional space. We introduce therefore the following notion of the size of the orthogonal complement to the biorthogonal system.

Definition 3.1. Let $\{k_\lambda\}$ be an exact system of reproducing kernels in a space $\mathcal{H} \in \mathfrak{R}$ with the generating function G, and let $\mathcal{S}$ be the corresponding space parameterizing the orthogonal complement. Let M be a positive increasing function on $\mathbb{R}_+$. We say that the orthogonal complement to the biorthogonal system has size M if there exists $S \in \mathcal{S}$ such that, for some $y_0 > 0$,

$$\log |S(\mathrm{i}y)| \geq M(|y|), \quad |y| > y_0. \qquad \square$$

From now on we will consider only the situation when $T = \{t_n\} \subset \mathbb{R}$. As we have mentioned in Section 1, this case corresponds to de Branges spaces. Thus, we assume that $\mathcal{H} = \mathcal{H}(E)$ and the generating function F of the sequence T is given by $F = \frac{E+E^*}{2}$. Our first theorem shows that under some mild restrictions on t_n any function in $\mathcal{S}$ is of zero exponential type, and so, for any $\varepsilon > 0$, the orthogonal complement cannot have the size $M(r) = \varepsilon r$.

Theorem 3.2. Let $t_n \in \mathbb{R}$. If G is the generating function of an exact system of reproducing kernels and $\mathcal{S}$ is the corresponding space, then any $S \in \mathcal{S}$ is of zero exponential type. $\square$

From this we have an immediate corollary.

Corollary 3.3. Let $t_n \in \mathbb{R}$. Then for any $\varepsilon > 0$ the orthogonal complement of a biorthogonal system cannot have the size $M(r) = \varepsilon r$. $\square$

In what follows we will use essentially the inner–outer factorization of H^2 functions (see, e.g., [13, 17]). Recall that a function f is said to be in the *Smirnov class* if $f = g/h$, where g and h are bounded analytic functions in $\mathbb{C}^+$ (or functions in H^p) and h is outer.

Proo of Theorem 3.2. If λ_0 is a zero of G, then $\frac{G}{z-\lambda_0} \in \mathcal{H}(E)$. Hence, $h := \frac{G}{(z-\lambda_0)E} \in H^2$. Let us show that h has no singular inner factor of the form $e^{2\mathrm{i}az}$, $a > 0$ (note that h has no other singular factors since it is analytic on $\mathbb{R}$). Indeed, if $e^{-2\mathrm{i}az} h \in H^2$, then put

$$H(z) = e^{-\mathrm{i}az} \frac{\sin az}{z} G(z).$$

Then $H/E \in H^2$ and also $H^*/E \in H^2$, so $H \in \mathcal{H}(E)$, which contradicts the fact that $\{\lambda : G(\lambda) = 0\}$ is a uniqueness set for $\mathcal{H}(E)$.

Consider the inner function $\Theta = E^*/E$. Then $2F = E(1+\Theta)$ and we have

$$\frac{G(z)S(z)}{E(z)} = \sum_m \frac{a_m}{\|k_{t_m}\|_{\mathcal{H}}} \cdot \frac{1+\Theta(z)}{t_m - z} G(t_m) =: f.$$

We show that the right-hand side function f is in the Smirnov class in $\mathbb{C}^+$. First of all note that if $v_m \ge 0$ and $\{v_m\} \in \ell^1$, then

$$\operatorname{Im} \sum_m \frac{v_m}{t_m - z} > 0, \quad z = x + iy \in \mathbb{C}^+,$$

and so this sum is in the Smirnov class. The same is obviously true also for an arbitrary sequence $\{v_m\} \in \ell^1$.

Since $\frac{G}{z-\lambda_0} \in \mathcal{H}(E)$ we have $\{\|k_{t_m}\|_{\mathcal{H}}^{-1} t_m^{-1} G(t_m)\} \in \ell^2$. Hence, $\{v_m\} \in \ell^1$ where

$$v_m = \frac{a_m}{\|k_{t_m}\|_{\mathcal{H}}} \cdot \frac{G(t_m)}{t_m},$$

and we have

$$\frac{f(z)}{1+\Theta(z)} = \sum_m v_m \frac{t_m}{t_m - z} = \sum_m v_m + z \sum_m \frac{v_m}{t_m - z}.$$

Hence f is in the Smirnov class.

Thus $S = f(\frac{G}{E})^{-1}$ is a ratio of two functions of bounded type, and so is a function of bounded type (zeros of Blaschke products cancel). Moreover, since S is analytic on $\mathbb{R}$ it is in the Smirnov class unless it has a factor of the form e^{-2iaz} in its canonical inner-outer factorization. However, it cannot happen, since, as we have seen, G/E has no singular inner factor.

By completely identical arguments, S is in the Smirnov class in the lower half-plane $\mathbb{C}^-$. Since S is in the Smirnov class both in $\mathbb{C}^+$ and in $\mathbb{C}^-$, it is of zero exponential type by Krein's theorem (see, e.g., [11, Chapter I, Section 6]). ■

As we have seen in Corollary 3.3, the linear growth of the function M (which determines the size of the orthogonal complement) is not possible. The following proposition, which applies to the case $\mathcal{H}(\mathbb{Z}, b)$ (that is, $t_n = n$, $n \in \mathbb{Z}$), provides a converse result: for any slower growth, the size M for the orthogonal complement may be achieved for some choice of b_n.

Proposition 3.4. Let $M(r)/r$ be a decreasing function which tends to zero when $r \to +\infty$. Then there exist a sequence b_n and an exact system $\{k_\lambda\}$ in the space of entire functions $\mathcal{H}$ (corresponding to the space $\mathcal{H}(\mathbb{Z}, b)$) such that the orthogonal complement to the biorthogonal system has size M. □

Proof. First of all, we "atomize" function M. There exists an increasing integer-valued function μ with jumps at some half-integer points such that $M - \mu \in L^\infty$. We will assume that $\mu(0) = 0$, $\mu(1) = 1$. Let $S(z) = \prod_{t \in \operatorname{supp} d\mu}(1 - \frac{z^2}{t^2})$. We want to estimate $|S(iy)|$ for large $|y|$:

$$\log|S(\mathrm{i}y)| = \int_0^\infty \log\left(1 + \frac{y^2}{t^2}\right) \mathrm{d}\mu(t) = 2y^2 \int_0^\infty \frac{\mu(t)\,\mathrm{d}t}{t(y^2 + t^2)}$$
$$\geq y\left(\inf_{t\in[1/2,y]} \frac{\mu(t)}{t}\right) \cdot \int_{1/2}^{y} \frac{y}{y^2 + t^2}\,\mathrm{d}t \gtrsim \mu(y), \quad |y| \to \infty.$$

Now we put $G(z) = \cos(\pi z)/S(z)$, $b_n = S^{-1}(n)$. The function G has an exponential type π and we can verify that G is the generating function of some exact system using the same arguments as in the proof of Example 1.3. Indeed, if $TG \in \mathcal{H}$, then T is of zero exponential type and $\sum_n |T(n)|^2 < +\infty$, so $T \equiv 0$. It only remains to note that $(z - 1/2)^{-1} S(z) \in \mathcal{S}$. ■

As we have seen in the proof of Proposition 3.4 the size of orthogonal complement to a biorthogonal system corresponds to the speed of decrease of the coefficients b_n. It becomes bigger when b_n decrease faster. Nevertheless from Theorem 1.4 we see that for extremely small b_n biorthogonal system has a finite codimension. This corresponds to polynomial size $M(r) = n \log r$.

Proof of Theorem 1.4. Suppose G is the generating function of some exact system, and let λ_0 be a zero of G. Hence,

$$\frac{G(z)}{(z - \lambda_0)F(z)} = \sum_n \frac{c_n}{z - t_n}, \quad \sum_n \frac{|c_n|^2}{b_n} < +\infty.$$

In particular, $\{c_n/b_n\} \in \ell^2(T, b)$. Since the polynomials are dense in $\ell^2(T, b)$, it follows that there exists $N \in \mathbb{N} \cup \{0\}$ such that $\langle \{c_n/b_n\}, t_n^N \rangle_{\ell^2(T,b)} = \sum_n c_n t_n^N \neq 0$. We take the smallest N

with this property. Let us estimate G/F from below on the imaginary axis. We have

$$\sum_n \frac{c_n}{\mathrm{i}y - t_n} = \sum_{|t_n|\le |y|/2} \frac{c_n}{\mathrm{i}y - t_n} + \sum_{|t_n|>|y|/2} \frac{c_n}{\mathrm{i}y - t_n}.$$

Since $\sum_n |c_n|\cdot|t_n|^k < \infty$ for any k, we have

$$\left|\sum_{|t_n|>|y|/2} \frac{c_n}{\mathrm{i}y - t_n}\right| = O\left(\frac{1}{y^{N+2}}\right), \quad |y| \to +\infty.$$

For $|t_n| \le |y|/2$, we have

$$(iy - t_n)^{-1} = \sum_{k\ge 0} \frac{t_n^k}{(\mathrm{i}y)^{k+1}} = \sum_{k=0}^{N} \frac{t_n^k}{(\mathrm{i}y)^{k+1}} + r_n(y)\frac{t_n^{N+1}}{(\mathrm{i}y)^{N+2}},$$

where $|r_n(y)| \le 2$. Hence,

$$\sum_{|t_n|\le|y|/2} \frac{c_n}{\mathrm{i}y - t_n} = \sum_{k=0}^{N} \frac{1}{(\mathrm{i}y)^{k+1}} \sum_n c_n t_n^k - \sum_{k=0}^{N} \frac{1}{(\mathrm{i}y)^{k+1}} \sum_{|t_n|>|y|/2} c_n t_n^k$$
$$+ \frac{1}{(\mathrm{i}y)^{N+2}} \sum_n c_n r_n(y) t_n^{N+1} = \frac{1}{(\mathrm{i}y)^{N+1}} \sum_n c_n t_n^N + O\left(\frac{1}{y^{N+2}}\right).$$

We conclude that $|G(\mathrm{i}y)|/|F(\mathrm{i}y)| \ge c|y|^{-N-1}$, $|y| \to +\infty$.

Now let $S \in \mathcal{S}$ and so, for some $\{a_m\} \in \ell^2$,

$$\frac{S(z)G(z)}{F(z)} = \sum_m \frac{a_m}{\|k_{t_m}\|_{\mathcal{H}}} \cdot \frac{G(t_m)}{t_m - z}.$$

We have

$$\left|\sum_m \frac{a_m}{\|k_{t_m}\|_{\mathcal{H}}} \cdot \frac{G(t_m)}{t_m - \mathrm{i}y}\right| \le \sum_m |a_m| \frac{|G(t_m)|}{\|k_{t_m}\|_{\mathcal{H}}|t_m|} =: A < +\infty,$$

since $\{\|k_{t_m}\|_{\mathcal{H}}^{-1}|t_m|^{-1}G(t_m)\} \in \ell^2$. Hence,

$$|S(iy)| \le A\frac{|F(\mathrm{i}y)|}{|G(\mathrm{i}y)|} \le C|y|^{N+1}, \quad |y| \to \infty.$$

By Theorem 3.2, S is of zero exponential type, and thus, a polynomial of degree at most $N+1$. Hence, $\mathcal{S}$ has a finite dimension. ■

Using the known results on density of polynomials we can give more examples of the situation where all biorthogonal systems have finite codimension. These examples deal with one-sided sequences with power growth.

Example 3.5.

(1) Let $t_n = n^{1/\beta}$, $n \in \mathbb{N}$, and let $b_n = \exp(-At_n^\alpha)$, where $A, \alpha > 0$. If $\beta \ge \frac{1}{2}$, then the polynomials are dense in $\ell^2(T, b)$ if and only if $\alpha \ge \frac{1}{2}$. If $\beta < \frac{1}{2}$, then the polynomials are dense in the space $\ell^2(T, b)$ for $\alpha > \beta$ and are not dense for $\alpha < \beta$; if $\alpha = \beta$, then the polynomials are dense if and only if $A \ge \pi \cot \pi\beta$ (see [7] for details).

(2) The situation changes when we add a sparser sequence on the negative semiaxis. Let $\beta > 1$ and let

$$t_n = \begin{cases} n^{1/\beta}, & n > 0,\ n \in \mathbb{Z}, \\ -|n|^{1/(\beta-1)}, & n < 0,\ n \in \mathbb{Z}. \end{cases}$$

Let $b_n = \exp(-t_n^\alpha)$, $\alpha > 0$. Using the results of [3], one can show that for $1 < \beta < \frac{3}{2}$, the polynomials are dense in $\ell^2(T, b)$ for $\alpha > \frac{1}{2}$ and not dense for $\alpha < \frac{1}{2}$. For $\frac{3}{2} \le \beta < 2$, the polynomials are dense in $\ell^2(T, b)$ for $\alpha > \beta - 1$ and not dense for $\alpha < \beta - 1$. □

4 Incomplete Biorthogonal Systems in Model Subspaces

We start with an analog of Theorem 1.6 for the unit disc $\mathbb{D}$. Let Θ be an inner function in $\mathbb{D}$ and, as in the half-plane case, let $K_\Theta = H^2 \ominus \Theta H^2$. Denote by $\sigma(\Theta)$ the boundary spectrum of Θ. A point $\zeta \in \mathbb{R}$ is said to be a *Carathéodory point* if Θ has an angular derivative at ζ, that is, there exist the nontangential limit $\Theta(\zeta)$ with $|\Theta(\zeta)| = 1$ as well as the nontangential limit $\Theta'(\zeta) = \lim_{z\to\zeta} \frac{\Theta(z)-\Theta(\zeta)}{z-\zeta}$. By the Ahern–Clark theorem [1], this is equivalent to the fact that the reproducing kernel k_ζ belongs to K_Θ and any element in K_Θ has a finite nontangential boundary value at ζ. Finally, if $\Theta = BI_\nu$ is a factorization of Θ into a Blaschke product and a singular inner function, then ζ is a Carathéodory point if and only if

$$|\Theta'(\zeta)| = \sum_n \frac{1 - |z_n|^2}{|\zeta - z_n|^2} + \int_{\mathbb{T}} \frac{d\nu(\tau)}{|\zeta - \tau|^2} < +\infty.$$

Here z_n are zeros of B and ν is a singular measure on the circle $\mathbb{T}$.

Theorem 4.1. Let Θ be an inner function in $\mathbb{D}$ such that there exists $\zeta \in \sigma(\Theta)$ which is a Carathéodory point. Then there exists an exact system of reproducing kernels such that its biorthogonal system is not complete. □

Without loss of generality we may assume that Θ is a Blaschke product. Indeed, we can always pass to a Frostman shift $B = \frac{\Theta-\gamma}{1-\bar{\gamma}\Theta}$, $|\gamma| < 1$, which is a Blaschke product, and the map $f \mapsto (1-|\gamma|^2)^{1/2}\frac{f}{1-\bar{\gamma}\Theta}$ is a unitary operator from K_Θ onto K_B, which maps the kernels to the kernels (up to constant bounded factors).

We also will move to the upper half-plane so that ζ goes to ∞. Recall that for a Blaschke product B, the property to have an "angular derivative at ∞" is equivalent to any of the following:

(a) there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that

$$\operatorname{Re}\frac{\alpha+\Theta(z)}{\alpha-\Theta(z)} = p\operatorname{Im} z + \frac{\operatorname{Im} z}{\pi}\int_{\mathbb{R}}\frac{d\mu(t)}{|t-z|^2}, \qquad z \in \mathbb{C}^+,$$

for a singular measure μ and $p > 0$;

(b) there exists a unimodular constant α such that $\alpha - B \in K_B$;

(c) there exist a unimodular α and $q > 0$ such that

$$1 - \bar{\alpha}B(\mathrm{i}y) = \frac{q}{y} + o\left(\frac{1}{y}\right), \qquad y \to +\infty.$$

In this case

$$q = 2p^{-1} = 2\sum_n y_n, \tag{4.1}$$

where $z_n = x_n + \mathrm{i}y_n$ are zeros of B (and, in particular, the series $\sum_n y_n$ converges). Of course we will assume $\alpha = 1$, so $1 - B \in K_B$. Note also that for any $g \in K_B$ there exists a finite limit $\lim_{y\to\infty} yg(\mathrm{i}y)$, and

$$(g, 1-B) = 2\pi \lim_{y\to\infty} yg(\mathrm{i}y).$$

In what follows we again use essentially the inner–outer factorization of H^2 functions. If $m \geq 0$ and $\log m \in L^2(\frac{dt}{t^2+1})$, then we denote by O_m the outer function with the modulus m on $\mathbb{R}$.

If we identify the functions in K_Θ and their boundary values on $\mathbb{R}$, then an equivalent definition of K_Θ is $K_\Theta = H^2 \cap \Theta\overline{H^2}$. Thus, we have a criterion for the inclusion

$f \in K_\Theta$ which we will repeatedly use:

$$f \in K_\Theta \Longleftrightarrow f \in H^2 \quad \text{and} \quad \Theta \bar{f} \in H^2. \tag{4.2}$$

In the following lemmas we always assume that Θ is an inner function such that ∞ is a Carathéodory point and $1 - \Theta \in K_\Theta$.

Our first lemma shows that the zeros of K_Θ function may be concentrated in the upper half-plane.

Lemma 4.2. Let $f = O_m B I \in K_\Theta$, where B is a Blaschke product and I is some inner function. Then there exists a function $g = O_{\tilde{m}} \tilde{B} \in K_\Theta$ such that $\Theta \bar{g} = O_{\tilde{m}}$ is outer, B divides the Blaschke product $\tilde{B}$, and $|O_{\tilde{m}}| \asymp |O_m|$.

Moreover, if $\lim_{y\to\infty} y f(\mathrm{i}y) = 0$, we can choose g so that $\lim_{y\to\infty} y g(\mathrm{i}y) = 0$. □

Proof. By the criterion (4.2), we have $\Theta \bar{f} \in H^2$, and hence, $\Theta \bar{f} = O_m J$ for some inner function J. Then, again by (4.2), the function $f_1 = O_m B I J$ is in K_Θ and $\Theta \bar{f_1} = O_m$. If IJ is a Blaschke product we are done. Assume now that $f_1 = O_m B_1 K$, where K is a singular inner function. Replace K by its Frostman shift $K_1 = \frac{K-\gamma}{1-\bar{\gamma}K}$, $|\gamma| < 1$, which is a Blaschke product. Put

$$g = O_m (1 - \bar{\gamma} K) K_1 B_1 = O_m (K - \gamma) B_1, \quad O_{\tilde{m}} = O_m (1 - \bar{\gamma} K).$$

Then $\Theta \bar{g} = \Theta \bar{O}_m (\bar{K} - \bar{\gamma}) \bar{B}_1 = O_m (1 - \bar{\gamma} K)$ since $\Theta \bar{O}_m \bar{B}_1 = K O_m$. Thus, $g \in K_\Theta$ and $|O_{\tilde{m}}| \asymp |O_m|$.

Finally, note that if $\lim_{y\to\infty} y f(\mathrm{i}y) = 0$, that is, f is orthogonal to $1 - \Theta$, then the same is true for f_1. Hence,

$$0 = (f_1, 1 - \Theta) = (\bar{\Theta} f_1, \bar{\Theta} - 1) = (\Theta - 1, \Theta \bar{f_1}) = (\Theta - 1, O_m).$$

Thus, $\lim_{y\to\infty} y O_m(\mathrm{i}y) = 0$, and the same is true for g, since $|O_{\tilde{m}}| \asymp |O_m|$. ■

The next lemma shows that one can get rid of real zeros of a function in K_Θ. Here, we will use a lemma due to Makarov and Poltoratski [15].

Lemma 4.3. Let f be a function in K_Θ, let $a_n \in \mathbb{R}$, $a_n < a_{n+1}$, $|a_n| \to \infty$, $n \to \infty$, and assume that there exist nonnegative integers m_n such that

$$(t-a_n)^{-m_n} f \in L^2(a_n-\delta_n, a_n+\delta_n)$$

for a small $\delta_n > 0$, but $(t-a_n)^{-m_n-1} f \notin L^2(a_n-\delta_n, a_n+\delta_n)$. Let $\{w_j\}$ be a sequence of points in $\mathbb{C}^+$ with $|w_j| \to \infty$. Then there exists a function $h \in K_\Theta$ such that f/h is locally bounded on each open interval (a_n, a_{n+1}), for any n we have $\frac{h}{z-a_n} \notin L^2(a_n-\delta_n, a_n+\delta_n)$, and

$$|h(w_j)| \gtrsim |f(w_j)|.$$

Moreover, if $\lim_{y\to\infty} yf(iy) = 0$, we can take h so that $\lim_{y\to\infty} yh(iy) = 0$. $\square$

Proof. Assume for a moment that all $m_n = 1$. Then we divide f by a function of the form $1-J$ where J is a meromorphic Blaschke product and $J = 1$ exactly at $\{a_n\}$. We also want to do this so that $f/(1-J)$ is still in H^2 (and hence in K_B). Such a choice of J is possible by [15, Lemma 3.15]. The function J should be constructed as

$$\frac{1+J(z)}{1-J(z)} = \frac{1}{i}\sum_n \frac{\nu_n}{a_n - z},$$

where $\nu_n > 0$ are very small (the decay depends on the norms of $(t-a_n)^{-m_n} f$ in $L^2(a_n - \delta_n, a_n+\delta_n)$). Since the right-hand side has a positive real part, J is a meromorphic inner function, and also,

$$\frac{2J(z)}{J(z)-1} = 1 + \frac{1}{2i}\sum_n \frac{\nu_n}{a_n - z}.$$

Then we have

$$h = \frac{2J}{J-1} f \in H^2.$$

Also, $\Theta\bar{h} = \Theta\bar{f}\cdot\frac{1}{1-J} \in H^2$, and so $h \in K_\Theta$ by (4.2). Obviously, f/h is locally bounded on each interval (a_n, a_{n+1}), Since $\lim_{y\to\infty}\frac{2J(iy)}{1-J(iy)} = -1$, we have $\lim yh(iy) = 0$. Finally, note that by choosing ν_n sufficiently small we can make $\frac{|2J(w_j)|}{|1-J(w_j)|}$ as close to 1 as we want.

In general case when $m_n \neq 1$ we repeat the procedure and construct a sequence of meromorphic inner functions J_k so that a_n is in the set $\{t : J_k(t) = 1\}$ exactly for m_n of the functions J_k. Put

$$h = g \prod_k \frac{2J_k}{J_k - 1}.$$

Choosing the masses ν_n^k in the definition of J_k sufficiently small we may achieve that the product converges to $h \in K_\Theta$, $\lim_{y\to\infty} yh(iy) = 0$ and $|h(w_j)| \geq c|f(w_j)|$ for a positive constant c. We omit the technicalities. ■

Lemma 4.4. Let f be a Smirnov class function in $\mathbb{C}^+$ such that f restricted to $\mathbb{R}$ has a Smirnov class extension to $\mathbb{C}^-$. Assume also that $\Delta \subset \mathbb{R}$ is an open interval such that f is in $L^2(\Delta)$. Then f has an analytic extension through Δ. □

Proof. Let $[x-r, x+r]$ be a subinterval of Δ. Since f is in the Smirnov class and in $L^2(\Delta)$ we have $f(\cdot + i\varepsilon) \to f$ in $L^2([x-r, x+r])$ when $\varepsilon \to +0$, and also when $\varepsilon \to -0$. Consider the contours $\gamma_+ = [x-r, x+r] \cup \{z \in \mathbb{C}^+ : |z-x| = r\}$ and $\gamma_- = [x-r, x+r] \cup \{z \in \mathbb{C}^- : |z-x| = r\}$ with the standard orientations. If $z_0 \in \mathbb{C}^+$, $|z_0 - x| < r$, we have

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_+} \frac{S(z)}{z - z_0}\,\mathrm{d}z, \quad 0 = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_-} \frac{S(z)}{z - z_0}\,\mathrm{d}z$$

(step a bit from $\mathbb{R}$, apply the Cauchy formula and then pass to the limit). Taking the sum, we conclude that

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{|z-x|=r} \frac{S(z)}{z - z_0}\,\mathrm{d}z, \quad |z-x| < r, \ z \notin \mathbb{R},$$

and, hence, f has an analytic extension in the whole disc $|z-x| < r$. ■

Proof of Theorems 1.6 and 4.1. As we have seen, it suffices to prove the theorem for the case where $\Theta = B$ is a Blaschke product such that $\infty \in \sigma(B)$, ∞ is a Carathéodory point and $1 - B \in K_B$. The symbol q has the same meaning as in (4.1).

The idea of the proof is to construct a function g in K_B such that

(i) $g = B_\Lambda O_m$ and $B\bar{g} = O_m$, O_m is an outer function in K_B;
(ii) $\Lambda' = \Lambda \cup \{\lambda_0\}$ is a uniqueness set for K_B for any $\lambda_0 \in \mathbb{C}^+ \setminus \Lambda$;
(iii) g is orthogonal to $1 - B$ which means that $\lim_{y\to\infty} y|g(iy)| = 0$.

If such g is constructed, then the system $(z - \lambda_0)\frac{g(z)}{z-\lambda}$, $\lambda \in \Lambda'$, is biorthogonal to a complete system $\{k_\lambda\}_{\lambda\in\Lambda'}$, but it is not itself complete, since it is orthogonal to $1 - B$.

We will consider separately two cases with slightly different proofs. Let us start with an easier case.

Case 1: Assume that

$$\limsup_{y\to+\infty} y^2 \left|1 - B(\mathrm{i}y) - \frac{q}{y}\right| = +\infty. \tag{4.3}$$

Consider the function

$$f(z) = 1 - B(z) - \frac{\mathrm{i}q}{z - \bar{z}_0},$$

where $z_0 = x_0 + \mathrm{i}y_0$ is some zero of B. Obviously, $(f, 1 - B) = \lim_{y\to\infty} y|f(\mathrm{i}y)| = 0$. Also note that for $t \in \mathbb{R}$,

$$f(t) = \frac{t - x_0 + \mathrm{i}y_0 - \mathrm{i}q}{t - x_0 + \mathrm{i}y_0} - B(t).$$

Since $q > 2y_0$ we have for almost all t

$$|f(t)| \ge \left|\frac{t - x_0 + \mathrm{i}y_0 - \mathrm{i}q}{t - x_0 + \mathrm{i}y_0}\right| - 1 \ge \frac{C}{t^2}. \tag{4.4}$$

Now applying Lemma 4.2 to f we obtain a function $g = O_m B_\Lambda \in K_B$ such that $|g| \asymp |f|$ on $\mathbb{R}$ and $B\bar{g} = O_m$.

Let us show that $\Lambda' = \Lambda \cup \{\lambda_0\}$ is a uniqueness set for K_B for any $\lambda_0 \in \mathbb{C}^+ \setminus \Lambda$. Assume the converse. Then there exists a function $F \in K_B$ which vanishes on Λ'. We can write $F = Sg$ for a function S which is analytic in $\mathbb{C}^+$. Then $S = (F/B_\Lambda)/O_m$ is in the Smirnov class in $\mathbb{C}^+$. Also, since $F \in K_B$ we have

$$B\bar{F} = B\bar{g}\bar{S} = O_m\bar{S} \in K_B.$$

So $S^*(z) = \overline{S(\bar{z})}$ has a Smirnov class extension to the upper half-plane, or S itself is a Smirnov class function in $\mathbb{C}^-$. Also, in view of (4.4), S is locally in L^2 on $\mathbb{R}$ and hence, by Lemma 4.4, S is entire. Since S is in the Smirnov class both in $\mathbb{C}^+$ and in $\mathbb{C}^-$, it is of zero exponential type by Krein's theorem (see, e.g., [11, Chapter I, Section 6]).

On the other hand, applying (4.3) once again we conclude that $S \in (t+i)^2 L^2(\mathbb{R})$, and so S is a polynomial of degree at most 1. On the other hand, we have $S^* O_m \in K_B$.

Note that

$$|O_m(\mathrm{i}y)| \geq |f(\mathrm{i}y)| = \left|1 - B(\mathrm{i}y) - \frac{q}{y} + O\left(\frac{1}{y^2}\right)\right|.$$

If S is not a constant, then, by (4.3), $\limsup |yS^*(iy)O_m(\mathrm{i}y)| = +\infty$, but this is impossible, since this limit is finite for any function in K_B.

Case 2: Now assume that (4.3) is not satisfied, that is, there exists $M > 0$ such that

$$y^2\left|1 - B(\mathrm{i}y) - \frac{q}{y}\right| \leq M \tag{4.5}$$

for sufficiently large y. The proof for Case 1 does not work, since if (4.5) is satisfied we cannot claim that S^*O_m is not in K_B when S is a polynomial of degree 1; we need to use the fact that ∞ is a point of the spectrum (i.e., a limit point for the zeros of B), since for finite Blaschke products the argument should fail. However, we will again construct g satisfying (i)–(iii).

Step 1: Take a very sparse subsequence $z_n = x_n + \mathrm{i}y_n$ of zeros of B so that $|z_n| \to \infty$ and $\{z_n\}$ is a Carleson interpolating sequence. Thus, the functions $\sqrt{y_n}(z - \bar{z}_n)^{-1}$ form a Riesz sequence in K_B (see, e.g., [17, Lecture VIII]). Also we may assume that $y_n < 1$, $x_n > 2M/q$ for all n, and $|x_n - x_k| > x_n/2$ for any $n, k, n \neq k$.

Now take a sequence $(c_n) \in \ell^2$, $c_n > 0$, such that

(a) $\sum_n c_n^2 x_n^2 = +\infty$;
(b) $\sum_n \sqrt{y_n} c_n = 1$ (recall that $\sum_n y_n < +\infty$);
(c) $qc_n > 4\sqrt{y_n}$.

It is easy to see that these conditions may be achieved, if y_n tends to zero sufficiently fast (e.g., if $\sum_n \sqrt{y_n} = m < 1$ take $c_n = 1/m$). It follows from (b) and the fact that $x_n > 2M/q$, that

(d) $q \sum \sqrt{y_n} c_n x_n > M$ (may be $+\infty$).

Put

$$f(z) = 1 - B(z) - \mathrm{i}q \sum \frac{\sqrt{y_n} c_n}{z - \bar{z}_n}.$$

Obviously, this is a function in K_B which is orthogonal to $1-B$. Also, for sufficiently large y,

$$y^2|\mathrm{Im}\, f(\mathrm{i}y)| = y^2\left|\mathrm{Im}\,(1-B(\mathrm{i}y)) - q\sum\frac{\sqrt{y_n}c_n x_n}{x_n^2+(y+y_n)^2}\right|$$

$$\geq y^2 q\sum\frac{\sqrt{y_n}c_n x_n}{x_n^2+(y+y_n)^2} - M \geq C > 0$$

by condition (d) and the fact that $y^2|\mathrm{Im}\,(1-B(\mathrm{i}y))| \leq M$.

We will see that $f(z_n) = -\frac{qc_n}{\sqrt{y_n}} + O(1)$ (since x_n tends to infinity very rapidly), so f has a good growth along z_n and we will see later that zf cannot be in H^2.

Step 2: Applying Lemma 4.2 we would obtain from f a new function $g \in K_B$ such that $B\bar{g}$ is outer. However, f may have real zeros and we shall divide them out first. Consider the function

$$R(t) = \left|1-\mathrm{i}q\sum\frac{\sqrt{y_n}c_n}{t-\bar{z}_n}\right|^2 - 1 = \left(1-\mathrm{i}q\sum\frac{\sqrt{y_n}c_n}{t-\bar{z}_n}\right)\left(1+\mathrm{i}q\sum\frac{\sqrt{y_n}c_n}{t-z_n}\right) - 1.$$

It is analytic on $\mathbb{R}$ which implies that zeros a_n of R are of finite multiplicities and $a_n \to \infty$. Hence, we can represent $\mathbb{R} = \bigcup[a_n, a_{n+1}]$ where $a_n < a_{n+1} \to \infty$ and R is locally separated from zero on the open intervals (a_n, a_{n+1}). Then

$$|f(t)| \geq \left|\left|1-\mathrm{i}q\sum\frac{\sqrt{y_n}c_n}{t-\bar{z}_n}\right| - 1\right|$$

is locally separated from 0 almost everywhere on each of the intervals (a_n, a_{n+1}).

By Lemma 4.3 there exists a function $h \in K_B$ such that $\frac{h}{z-a_n} \notin L^2(a_n-\delta, a_n+\delta)$ for any $\delta > 0$, $|h/f|$ is locally separated from 0 on any interval (a_n, a_{n+1}) and $|h(z_n)| \gtrsim |f(z_n)|$. Now we apply Lemma 4.4 to the function h and obtain a function $g = O_m B_\Lambda \in K_B$ such that $B\bar{g} = O_m$ is outer, g is separated from zero locally on the intervals (a_n, a_{n+1}), and $\lim_{y\to\infty} yg(\mathrm{i}y) = 0$. Also we have

$$|O_m(z_n)| \asymp |O_{|h|}(z_n)| \geq |h(z_n)| \gtrsim |f(z_n)|.$$

Step 3: To complete the proof we need to show that $\Lambda \cup \{\lambda_0\}$ is a uniqueness set for K_B. Assume the converse. Then $F = Sg \in K_B$ for some function S analytic in $\mathbb{C}^+$ which vanishes at λ_0. As in the proof of Case 1, S is in the Smirnov class both in $\mathbb{C}^+$ and in $\mathbb{C}^-$. Also $S = F/g$ is locally in L^2 on (a_n, a_{n+1}). By Lemma 4.4 S is meromorphic with possible poles in a_n. Since R has zeros of finite multiplicities, there exists m_n such that near a_n we have

$$|f(t)| \geq C|R(t)| \geq C|t-a_n|^{m_n}$$

for some $C>0$, and an analogous estimate holds for g. Hence, S may have only poles at the points a_n. However, $(t-a_n)^{-1}g \notin L^2(a_n-\delta, a_n+\delta)$, and we conclude that S is an entire function which is of zero exponential type by Krein's theorem.

We have $|O_m(\mathrm{i}y)| \gtrsim |f(\mathrm{i}y)| \geq Cy^{-2}$, $y \to +\infty$. So,

$$|S(\mathrm{i}y)| \leq \left|\frac{F(\mathrm{i}y)}{B_\Lambda(\mathrm{i}y)}\right| |O_m(\mathrm{i}y)|^{-1} \leq Cy^{3/2}$$

and

$$|S^*(\mathrm{i}y)| \leq \frac{|(B\bar{F})(\mathrm{i}y)|}{|O_m(\mathrm{i}y)|} \leq Cy^{3/2}, \quad y \to +\infty.$$

We conclude that S is a polynomial of degree at most 1.

Step 4: To finish the proof of completeness of $\{k_\lambda\}_{\lambda\in\Lambda\cup\{\lambda_0\}}$, we need to show that S is a constant and, thus, zero. Note that we have $S^*O_m \in K_B$. Let S be a polynomial of degree 1. Then we have $zO_m \in H^2$, and so, since z_n is a Carleson sequence, we have

$$\sum_n y_n|z_n|^2|f(z_n)|^2 \leq C\sum_n y_n|z_n|^2|O_m(z_n)|^2 < +\infty. \tag{4.6}$$

On the other hand,

$$\operatorname{Re} f(z_n) = 1 - \frac{qc_n}{2\sqrt{y_n}} - q\sum_{k\neq n}\frac{(y_n+y_k)\sqrt{y_k}c_k}{|z_n-\bar{z}_k|^2} = 1 - \frac{qc_n}{2\sqrt{y_n}} - d_n.$$

Using the properties of z_n we get

$$|d_n| = q\left|\sum_{k\neq n}\frac{\sqrt{y_k}c_k(y_n+y_k)}{|z_n-\bar{z}_k|^2}\right| \leq \frac{4q}{x_n}\sum_{k\neq n}\sqrt{y_k}c_k,$$

and hence $\sum_n y_n|z_n|^2|d_n|^2 < +\infty$. Recall also that by (c) we have $\frac{qc_n}{2\sqrt{y_n}} > 2$. Thus

$$|z_n|\cdot|\operatorname{Re} f(z_n)| + |z_nd_n| \geq \frac{qc_n|z_n|}{4\sqrt{y_n}},$$

but, by the choice of c_n,

$$\sum_n |z_n|^2 |c_n|^2 = +\infty,$$

so $\sum_n y_n |z_n|^2 |\mathrm{Re}\, f(z_n)|^2 = +\infty$, a contradiction to (4.6). Thus $S \equiv$ const. and, since $S(\lambda_0) = 0$, we finally conclude that $S \equiv 0$. ■

5 Sufficient Conditions for the Completeness

In this section, we discuss conditions sufficient for the completeness of the system biorthogonal to an exact system of reproducing kernels. Since the model spaces generated by meromorphic inner functions essentially coincide with de Branges spaces, we may obtain completeness for a class of model subspaces as a corollary of Theorem 1.2. However, it seems that this method does not work for the general model spaces.

On the other hand, even if we cannot say that *any* system biorthogonal to an exact system of reproducing kernels is complete, one may look for criteria for completeness of the biorthogonal system in terms of the generating function. Results of this type may have applications in the spectral theory, since systems biorthogonal to systems of reproducing kernels appear, for example, as eigenfunctions of certain rank 1 perturbations of unbounded selfadjoint operators [4].

The crucial property of the model spaces which makes it possible to apply the ideas from the proof of Theorem 1.2 is the representation of the functions in K_Θ via the so-called Clark measures [9], which is similar to (1.4). For any $\alpha \in \mathbb{C}$, $|\alpha| = 1$, the function $(\alpha + \Theta)/(\alpha - \Theta)$ has a positive real part in the upper half-plane. Hence, there exist $p_\alpha \geq 0$ and a measure μ_α with $\int (1+t^2)^{-1}\, \mathrm{d}\mu_\alpha(t) < +\infty$ such that

$$\mathrm{Re}\, \frac{\alpha + \Theta(z)}{\alpha - \Theta(z)} = p_\alpha \mathrm{Im}\, z + \frac{\mathrm{Im}\, z}{\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\sigma_\alpha(t)}{|t-z|^2}, \quad z \in \mathbb{C}^+.$$

The Clark theorem states that if μ_α is purely atomic (that is, $\mu_\alpha = \sum_n c_n \delta_{t_n}$, where δ_x denotes the Dirac measure at the point x) and $p_\alpha = 0$, then the system $\{k_{t_n}\}$ of reproducing kernels is an orthogonal basis in K_Θ. In the general case, if $p_\alpha = 0$, then the mapping

$$(C_\mu g)(z) = (\alpha - \Theta(z)) \int \frac{g(t)}{t-z}\, \mathrm{d}\mu_\alpha(t) \tag{5.1}$$

is a unitary map from $L^2(\mu)$ onto K_Θ. By a theorem of Poltoratski [20], any function $f \in K_\Theta$ has nontangential boundary values μ_α-a.e. for any α.

Note also that the following are equivalent:

(a) $p_\alpha > 0$ for some α;
(b) $\alpha - \Theta \in H^2(\mathbb{R})$ (i.e., ∞ is a Carathéodory point for Θ);
(c) $\mu_\beta(\mathbb{R}) < +\infty$ for some $\beta \in \mathbb{C}$, $|\beta| = 1$.

To see the equivalence of (a) and (c) note that if $\mu_\beta(\mathbb{R}) < +\infty$ and $p_\beta = 0$, then $p_{-\beta} > 0$.

For the model spaces generated by meromorphic inner functions we have the following immediate corollary of Theorem 1.2 which applies to the so-called *tempered inner functions*, that is inner functions such that $|(\arg\Theta)'(t)| \lesssim |t|^N$ for some $N > 0$.

Theorem 5.1. Let Θ be a meromorphic inner function. Write $\Theta = \exp(2i\varphi)$ on $\mathbb{R}$, where φ is a smooth increasing function on $\mathbb{R}$. Let $\{t_n\} = \{t \in \mathbb{R} : \Theta(t) = -1\}$. If $\alpha - \Theta \notin L^2(\mathbb{R})$ for any $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and for some $N > 0$ and $C > 0$,

$$\varphi'(t_n) \le C(|t_n| + 1)^N,$$

then any system biorthogonal to an exact system of reproducing kernels is complete in K_Θ. □

Proof. We may write $\Theta = E^*/E$ for a Hermite–Biehler function E. Then the mapping $f \mapsto Ef$ is a unitary map of K_Θ onto $\mathcal{H}(E)$. The function $A = \frac{E+E^*}{2} = \frac{E(1+\Theta)}{2}$ is the generating function for t_n, and the functions $k_{t_n}(z) = \frac{E(t_n)}{\pi i} \cdot \frac{A(z)}{z-t_n}$ form an orthogonal basis of reproducing kernels in $\mathcal{H}(E)$ (see formula (1.5)). By (1.6), $b_n = (\pi\varphi'(t_n))^{-1}$. Also note that

$$\sum_n b_n \delta_{t_n} = \sum_n \frac{B(t_n)}{\pi A'(t_n)} \delta_{t_n}$$

is the Clark measure μ_{-1} for Θ. Since $\alpha - \Theta \notin L^2(\mathbb{R})$ for any α we conclude that $\sum_n b_n = +\infty$, and the result follows from Theorem 1.2. ■

Remark 5.2. Theorem 5.1 may be extended to the case when the spectrum of Θ is a finite set and φ' has at most power growth near each of these points, that is, $\varphi'(t) \lesssim |t-a|^{-N}$ near $a \in \sigma(\Theta)$. However, it is not clear how to deal with the case when $\sigma(\Theta)$ has nonempty interior or when Clark measures have singular continuous parts. □

The following theorem applies to general inner functions. We now impose the restrictions on the generating function G in place of Θ. Recall that $G \in (t+i)H^2$. The following result shows that if G is sufficiently regular, that is, has at most power growth, then the biorthogonal system is complete.

Theorem 5.3. Let Θ be an inner function such that $1-\Theta \notin L^2(\mathbb{R})$ and $\mu_1(\mathbb{R}) = +\infty$ (these conditions are fulfilled, for example, if ∞ is not a Carathéodory point for Θ). Let G be the generating function of an exact system of reproducing kernels $\{k_{\lambda_n}\}$, $\lambda_n \in \mathbb{C}^+$. If for some $N > 0$ and $C > 0$

$$|G(t)| \le C|t+i|^N \quad \mu_1\text{-a.e.},$$

then the system $\frac{G(z)}{z-\lambda_n}$ is complete. □

Lemma 5.4. Let G be the generating function of an exact system of reproducing kernels $\{k_{\lambda_n}\}$, $\lambda_n \in \mathbb{C}^+$. Then $\Theta\bar{G}$ is an outer function and $\frac{G}{z-x} \notin L^2(\mathbb{R})$ for any $x \in \mathbb{R}$. □

Proof. Fix a zero λ_0 of G. Then we can write $G = (z-\lambda_0)g$ where $g \in K_\Theta$ and g vanishes on $\{\lambda_n\}_{n\neq 0}$. Then $\Theta\bar{G} = (t-\bar{\lambda}_0)\Theta\bar{g}$ on $\mathbb{R}$. Since $g \in K_\Theta$ we have $\Theta\bar{g} \in K_\Theta$ and we may write $\Theta\bar{g} = IO_m$. As in the proof of Lemma 4.2, if I is not a Blaschke product we may replace it by its Frostman shift $I_1 = \frac{I-\gamma}{1-\bar{\gamma}I}$, $|\gamma| < 1$, which is a Blaschke product. Then $h = g(1-\bar{\gamma}I)I_1$ is in K_Θ. If $I \neq 1$, then I_1 is a nontrivial Blaschke product, and so h vanishes on $\{\lambda_n\}_{n\neq 0}$ and on the zero set of I_1. This contradicts the completeness of $\{k_{\lambda_n}\}$.

If for some $x \in \mathbb{R}$, $\frac{G}{z-x} \in L^2(\mathbb{R})$, then by (4.2) $\frac{G}{z-x}$ is in K_Θ and vanishes on $\{\lambda_n\}$. ■

Proof of Theorem 5.3. Let $\mu = \mu_1$. Since $1-\Theta \notin L^2(\mathbb{R})$, we have representation (5.1) with $\alpha = 1$ for the elements of K_Θ. Let $h \in K_\Theta$ be orthogonal to all functions $\frac{G(z)}{z-\lambda_n}$. Then, for any n,

$$\left\langle \frac{G}{z-\lambda_n}, h \right\rangle_{H^2} = \left\langle \frac{G}{z-\lambda_n}, h \right\rangle_{L^2(\mu)} = \int \frac{G(t)\overline{h(t)}}{t-\lambda_n}\, d\mu(t) = 0. \tag{5.2}$$

Consider the function

$$L(z) = (1-\Theta(z)) \int \frac{G(t)\overline{h(t)}}{t-z}\, d\mu(t), \tag{5.3}$$

it is analytic in $\mathbb{C}^+$ and vanishes at λ_n. Hence, we may write $L = SG$ for a function S analytic in $\mathbb{C}^+$,

$$S(z)G(z) = (1-\Theta(z)) \int \frac{G(t)\overline{h(t)}}{t-z}\, d\mu(t). \tag{5.4}$$

We denote by $\mathcal{S}$ the linear space of all functions S for which (5.4) holds with some $h \in L^2(\mu)$.

Note that S has nontangential boundary values μ-a.e. and $G(t)S(t) = G(t)\overline{h(t)}$ μ-a.e. Indeed, G is locally bounded, and so $G\bar{h}\chi_{(-r,r)} \in L^2(\mu)$ for any $r > 0$ (by χ_E we denote the characteristic function of a set E). Write

$$L(z) = (1 - \Theta(z)) \int_{(-r,r)} \frac{G(t)\overline{h(t)}}{t-z}\,\mathrm{d}\mu(t) + (1 - \Theta(z)) \int_{\mathbb{R}\setminus(-r,r)} \frac{G(t)\overline{h(t)}}{t-z}\,\mathrm{d}\mu(t). \tag{5.5}$$

Then the first term is in K_Θ and has nontangential boundary values $G(x)\overline{h(x)}$ for $|x| < r$ μ-a.e. The second term is analytic for $|z| < r$ and tends to zero for a fixed z when $r \to \infty$. Hence, $S(t)G(t) = G(t)\overline{h(t)}$ μ-a.e.

By the arguments similar to Lemma 2.3 it is easy to show that for any $w \in \mathbb{C}^+$ and $S \in \mathcal{S}$ we have $\frac{S(z)-S(w)}{z-w} \in \mathcal{S}$ (just replace the sum by the integral with respect to μ).

Assume first that $\mathcal{S}$ is infinite dimensional. Then, as in the proof of Theorem 1.2, there exists $S \in \mathcal{S}$ with at least N zeros $w_1, \dots, w_N \in \mathbb{C}^+$ different from the points $\{\lambda_n\}$, and $T(z) := \frac{S(z)}{\prod_{l=1}^N (z-w_l)} \in \mathcal{S}$. For some $h \in L^2(\mu)$ we have

$$G(z)S(z) = (1 - \Theta(z)) \int \frac{G(t)\overline{h(t)}}{t-z}\,\mathrm{d}\mu(t),$$

and so,

$$G(z)T(z) = (1 - \Theta(z)) \int \frac{G(t)}{\prod_{l=1}^N (t-w_l)} \cdot \frac{\overline{h(t)}}{t-z}\,\mathrm{d}\mu(t).$$

Since $|G(t)| \lesssim |t+i|^N$ μ-a.e., we conclude that $\frac{G(t)}{\prod_{l=1}^N (t-w_l)} \cdot \overline{h(t)} \in L^2(\mu)$. Thus, for the function GT we have a representation of the form (5.1), and so $GT \in K_\Theta$. This contradicts the completeness of $\{k_{\lambda_n}\}$.

Now assume that $\mathcal{S}$ is finite dimensional. Since the transform $(\mathcal{D}_w S)(z) = \frac{S(z)-S(w)}{z-w}$ preserves the class $\mathcal{S}$, the functions $S, \mathcal{D}_w S, \mathcal{D}_w^2 S, \dots, \mathcal{D}_w^N S$ are linearly dependent for large N. We conclude that $\mathcal{S}$ consists of rational functions. Write $S = P/Q$, where P and Q are polynomials without common zeros. Since S is analytic in $\mathbb{C}^+$, Q has no zeros in $\mathbb{C}^+$. It follows from (5.5) that L is locally in L^2 on $\mathbb{R}$. If S has a pole x on $\mathbb{R}$ it follows that $\frac{G(z)}{z-x}$ is in $L^2(x-\delta, x+\delta)$ which contradicts Lemma 5.4. Assume, finally, that Q has a zero in $\mathbb{C}^-$. We have

$$\Theta(z)\,\overline{G(\bar{z})}\,\overline{S(\bar{z})} = (\Theta(z) - 1) \int \frac{\overline{G(t)}h(t)}{t-z}\,\mathrm{d}\mu(t).$$

The right-hand side is analytic in $\mathbb{C}^+$ whereas, by Lemma 5.4, $\Theta(z)\overline{G(\bar{z})}$ is an outer function in $\mathbb{C}^+$. Hence $\overline{S(\bar{z})}$ is analytic in $\mathbb{C}^+$ and, thus, S has no poles in $\mathbb{C}^-$. We conclude that S is a polynomial.

Without loss of generality we may assume that $S \equiv 1$. To complete the proof recall that $S(t)G(t) = G(t)\overline{h(t)}$ μ-a.e. Put $E_1 := \{t : G(t) \neq 0\}$, $E_2 := \mathbb{R} \setminus E_1$. We have $h(t) = 1$ μ-a.e. on E_1. Since $h \in L^2(\mu)$, but $\mu(\mathbb{R}) = +\infty$, we conclude that $\mu(E_1) < +\infty$ and $\mu(E_2) = +\infty$. Then the functions $h \in K_\Theta$ such that $h = 0$ μ-a.e. on E_1 form an infinite-dimensional subspace X of K_Θ. For any $h \in X$ we have $Gh = 0$ μ-a.e. and hence (5.2) holds. Since X is infinite dimensional and contains $\frac{h(z)}{z-\lambda}$ whenever $\lambda \in \mathbb{C}^+$, $h(\lambda) = 0$, there exists a nonzero function $h_0 \in X$ such that $(t+i)^N h_0 \in L^2(\mu)$. Then for L_0 defined by (5.3) with h_0 in place of h we have

$$L_0(z) = (1 - \Theta(z)) \int \frac{G(t)\overline{h_0(t)}}{t - z} \, d\mu(t) = (1 - \Theta(z)) \int \frac{G(t)}{(t+i)^N} \frac{(t+i)^N \overline{h_0(t)}}{t - z} \, d\mu(t).$$

The function L_0 vanishes at λ_n and belongs to K_Θ since it has a representation of the form (5.1). Hence, $L_0 \equiv 0$ and $h_0 = 0$ μ-a.e. This contradiction proves the theorem. ∎

Remark 5.5. If $G \in L^\infty(\mathbb{R})$, then the conditions $1 - \Theta \notin L^2(\mathbb{R})$ and $\mu_1(\mathbb{R}) = +\infty$ may be omitted. Indeed, $p_\alpha = 0$ for all unimodular α except at most one, and the functions in K_Θ admit representation (5.1). Then any function L defined by (5.3) is in K_Θ (note that $G\bar{h} \in L^2(\mu)$) and vanishes at λ_n. Hence, $h \equiv 0$ and we conclude that the biorthogonal system is complete. □

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