

A Proof of Factorization Formula for Critical Percolation

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Abstract: We give mathematical proofs to a number of statements which appeared in the series of papers by Simmons et al. (Phys Rev E 76(4):041106, 2007; J Stat Mech Theory Exp 2009(2):P02067, 33, 2009) where they computed the probabilities of several percolation events.

1. Introduction

Two-dimensional critical percolation has been extensively studied in the last two decades, both from a Conformal Field Theory point of view and by means of Schramm-Loewner evolution. The first approach provides a way to obtain explicit formulas for correlation functions (connection probabilities), whereas the second one, in particular, gives a key to rigorous proof of such results. The proof of Cardy's formula was the major step in Smirnov's proof of conformal invariance of percolation on the triangular lattice [12]. Lawler, Schramm and Werner [6] proved an asymptotical formula for the probability of an interval on the boundary of the domain to be connected to a small neighborhood of a bulk point. Later, Watts' formula for the simultaneous occurrence of up-down and left-right crossing in a rectangle was proven [2, 9]. A formula for the expected number of clusters separating two sides of a rectangle was established in [3], although the proof did not involve SLE techniques. We refer the reader to [4] for the introduction to the SLE topic.

In [11] Simmons et al. computed the density of the probability that there is a percolation cluster which connects a boundary interval with a boundary point and an interior point. In the present paper we provide a precise formulation and a proof of this result. This also leads to a rigorous proof of the exact factorization formula for a certain four-point correlation function proposed by the same authors.

Throughout the paper by percolation we mean the scaling limit of the critical percolation on the triangular lattice. We work only with percolation interfaces (boundaries of clusters) which by [12] have the law of $SLE(6)$. We remark here that this is only proven

for the site percolation on triangular lattice, but is conjectured to be true for other lattices as well.

Let $u_1 < u_2 < u_3$ be three points on the real line and w be a point in the upper half-plane \mathbb{H} . We are going to compute the asymptotic behavior of the probability that the percolation cluster attached to a boundary interval $[u_1, u_2]$ approaches the small neighborhood of u_3 and w . We will understand those neighborhoods in the following sense which is well-suited for SLE computations. Let K be the percolation cluster attached to $[u_1, u_2]$, by $r(w)$ we denote the conformal radius of the component of $\mathbb{H} \setminus K$ which contains w as seen from w . For a boundary point u_3 , we set $r(u_3) := \text{dist}(K \cap \mathbb{R}, u_3)$.

Denote by $\tilde{F}(u_1, u_2, u_3, w, s_1, s_2)$ the probability that $r(w) < e^{-s_1}$ and $r(u_3) < e^{-s_2}$. We can write

$$\tilde{F}(u_1, u_2, u_3, w, s_1, s_2) = C(u_1, u_2, u_3, s_2)F(u_1, u_2, u_3, w, s_1, s_2),$$

where $C(u_1, u_2, u_3, s_2)$ is the probability that $r(u_3) < e^{-s_2}$ and $F(u_1, u_2, u_3, w, s_1, s_2)$ is the conditional probability that $r(w) < e^{-s_1}$ given that $r(u_3) < e^{-s_2}$. We also introduce the following notation:

$$C(u_1, u_2, u_3) := \lim_{s \rightarrow \infty} e^{\frac{1}{3}s} C(u_1, u_2, u_3, s), \tag{1}$$

$$F(u_1, u_2, u_3, w, s_1) := \lim_{s_2 \rightarrow \infty} F(u_1, u_2, u_3, w, s_1, s_2), \tag{2}$$

$$F(u_1, u_2, u_3, w) := \lim_{s_1 \rightarrow \infty} e^{\frac{5}{48}s_1} F(u_1, u_2, u_3, w, s_1). \tag{3}$$

The existence of these limits will be clear from what follows. Denote $\mathbb{S} = \{0 < \text{Re } z < \infty, 0 < \text{Im } z < 1\}$. The main result of this paper is the following:

Theorem 1.1. *One has*

$$F(u_1, u_2, u_3, w) = K_5 |\psi'_{u_1, u_2, u_3}(w)|^{5/48} G(\text{Re } \psi_{u_1, u_2, u_3}(w), \text{Im } \psi_{u_1, u_2, u_3}(w)), \tag{4}$$

where ψ_{u_1, u_2, u_3} is the conformal map that transforms $\{\mathbb{H}, u_1, u_2, u_3\}$ to $\{\mathbb{S}, i, 0, \infty\}$, K_5 is a constant given by (19) and G is an explicit function given by (14)

We emphasize that the function $\tilde{F}(u_1, u_2, u_3, w, s_1, s_2)$ is a probability in *continuous* percolation, that is, the scaling limit of corresponding probability for percolation on the triangular lattice. Hence $F(u_1, u_2, u_3, w)$ is a double limit: we first take the mesh size to zero for fixed values of s_1, s_2 , and then take those to zero. A more natural formulation would concern the scaling behaviour of probabilities that a particular boundary site, interior site and a boundary segment are touched by the same cluster. We believe the result to be true in this formulation as well, but technical difficulties seem insuperable.

Thanks to conformal invariance of critical percolation, Theorem 1.1 has the following corollary: if Ω is a simply-connected domain with marked points u_1, u_2, u_3 on the boundary, w inside, and $\partial\Omega$ is smooth near u_3 , then

$$F_\Omega(u_1, u_2, u_3, w) = |\varphi'(u_3)|^{1/3} |\varphi'(w)|^{5/48} F(\varphi(u_1), \varphi(u_2), \varphi(u_3), \varphi(w)),$$

where φ is a conformal map from Ω to \mathbb{H} and F_Ω is the probability for the percolation in Ω that u_3 is connected to (u_1, u_2) and w , defined by the same limiting procedure as F .

To prove Theorem 1.1, we proceed as follows. In Sect. 2, we recall the formulae for three- and four-point functions for percolation. In Sect. 3, we prove that $SLE(6)$ started from u_2 and conditioned to hit u_3 and not to swallow u_1 has the law of $SLE(6, 2, -2)$.

Based on this, we write down certain PDE's for $F(u_1, u_2, u_3, w)$ and FC , the latter being the same as obtained in [11]. Finally, we prove that these PDE's together with boundary conditions determine F uniquely. In Sect. 4 we use our result to prove a factorization formula (20) that expresses the probability density of two points on the boundary of the domain to be connected to a bulk point in terms of pairwise connection probabilities.

For the next two sections, we will assume that all functions are sufficiently smooth, so that the forthcoming Itô calculus is legal. Surpassing this assumption is quite standard: after getting an explicit answer, which turns out to be a smooth function, one plugs that answer into the previous computations to prove that it is indeed a martingale, and then applies the optional stopping theorem. This is explained in detail for the function F in Sect. 3.

2. Three- and Four-Point Functions

The function $C(u_1, u_2, u_3, s)$ is the probability that there is a percolation crossing connecting $[u_1, u_2]$ with $[u_3 - e^{-s}, u_3 + e^{-s}]$, or, in terms of interfaces, the probability that the $SLE(6)$ curve started from u_2 touches the interval $[u_3 - e^{-s}, u_3 + e^{-s}]$ before swallowing u_1 . Hence $C(u_1, u_2, u_3, s)$ is given by Cardy's formula. Taking the limit as $s \rightarrow \infty$, one gets the following result:

Lemma 2.1. *As $s \rightarrow \infty$, one has*

$$C(u_1, u_2, u_3, s) \sim K_3 e^{-s/3} \left(\frac{u_2 - u_1}{(u_3 - u_2)(u_3 - u_1)} \right)^{1/3}, \quad \text{where}$$

$$K_3 = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{3})\Gamma(\frac{7}{6})}.$$

Hereinafter the notation $A \sim B$ means that the ratio of the two sides tends to one.

Proof. Theorem 3.2 in [5] proves this lemma when $u_1, u_2, u_3, e^{-s} = 1, \infty, x/2, x/2$. The statement of the lemma is obtained by conformal mapping to this case. \square

Remark 1. The rate of decay $e^{-s/3}$ of C is consistent with the value $1/3$ of the boundary one-arm exponent. Note that our event is similar to the one-arm event, the difference is that we condition the cluster to hit the interval $[u_1, u_2]$.

Remark 2. One can check by direct computation that the function $C(u_1, u_2, u_3)$ from Lemma 2.1 satisfies the following equation:

$$\left(\frac{2}{u_1 - u_2} \partial_{u_1} + \frac{2}{u_3 - u_2} \partial_{u_3} + 3\partial_{u_2}^2 - \frac{2}{3(u_3 - u_2)^2} \right) C = 0. \tag{5}$$

This equation means that C is a martingale which is conformally $1/3$ -covariant at u_3 and invariant at u_1 and u_2 , that is, $|g'_t(u_3)|^{1/3} C(g_t(u_1), B_{6t}, g_t(u_3))$ is a martingale, where g_t is the Loewner map driven by the Brownian motion B_{6t} started from u_2 .

Now we would like to prove an analog of this lemma when the point u_3 is inside the domain. Let $F(u_1, u_2, w, s)$ be the probability that $r(w) < e^{-s}$.

Lemma 2.2. *As $s \rightarrow \infty$, one has*

$$F(u_1, u_2, w, s) \sim K_4 e^{-\frac{5}{48}s} |\phi'(w)|^{5/48} (\sin(\pi\omega/2))^{1/3}, \tag{6}$$

where ω is the harmonic measure of (u_1, u_2) seen from w ; ϕ is a conformal map from \mathbb{H} to the unit disc, $\phi(w) = 0$, and

$$K_4 = \frac{18}{5\pi}.$$

Proof. Lawler, Schramm and Werner [6] proved a slightly weaker statement. Here we refine their result using spectral theory techniques.

Using locality of SLE(6), we reformulate the problem in terms of radial SLE(6). Let $h(\theta, s)$ denote the probability that the conformal radius of the complement of percolation cluster in the unit disc attached to a boundary arc of length θ is less than e^{-s} . Then it is proven in [6] that

$$c_1(\sin(\theta/4))^{1/3} e^{-5/48s} \leq h(\theta, s) \leq c_2(\sin(\theta/4))^{1/3} e^{-5/48s}, \tag{7}$$

and that the function h satisfies the PDE:

$$\partial_s h = (3\partial_{\theta\theta} + \cot(\theta/2)\partial_{\theta})h \tag{8}$$

with boundary conditions $h(\theta, 0) \equiv 1$, $h(0, s) \equiv 0$, and (a weak form of) the Neumann boundary condition “ $\partial_{\theta}h(\theta, s) \equiv 0$ ” at $\theta = 2\pi$. Write $f(\theta, s) := (\sin(\theta/2))^{1/3} h(\theta, s)$. Then (8) implies

$$\partial_s f = (3\partial_{\theta\theta} + 3V)f, \tag{9}$$

where $V(\theta) = \frac{1}{12} + \frac{1}{18} \cot^2(\frac{\theta}{2})$. Our goal is to define a self-adjoint operator corresponding to the right-hand side of (9) and to prove that it has a discrete spectrum, hence the function f for large s behaves like the leading eigenfunction. The proof below (and perhaps the result itself) is very standard in spectral theory. First of all, we consider the operator $\partial_{\theta\theta} + V$ acting on smooth functions with support inside $(0, 2\pi)$ and write explicitly the domain $D(\Lambda_0)$ of the L_2 -closure of this operator. Self-adjoint extensions of this closure are obtained by adding two-dimensional subspaces to $D(\Lambda_0)$. Different extensions correspond to different boundary conditions, and we consider two such extensions: Λ (which corresponds to the boundary conditions we are interested in) and Λ_1 (which corresponds to Dirichlet boundary conditions for h on both sides of $(0, 2\pi)$). It is a well-known fact that the spectral properties of Λ and Λ_1 are similar, so it is sufficient to prove that Λ_1 has a discrete spectrum. We reformulate this property in terms of quadratic forms, and then use Hardy’s inequality to prove that the form corresponding to Λ_1 is comparable to the form corresponding to $\partial_{\theta\theta}$, which is well-known to have a discrete spectrum.

Set $\Lambda_0 := (\partial_{\theta\theta} + V)$ with

$$\begin{aligned} D(\Lambda_0) &:= \\ &= \{f \in AC^1([0, 2\pi]) : \Lambda_0 f \in L_2(0, 2\pi), f(0) = f(2\pi) = f'(0) = f'(2\pi) = 0\} \\ &= \{f \in AC^1([0, 2\pi]) : f'' \in L_2(0, 2\pi), f(0) = f(2\pi) = f'(0) = f'(2\pi) = 0\}. \end{aligned}$$

The operator Λ_0 is closed in L_2 . Let ξ be a smooth decreasing function with $\xi \equiv 1$ on $[0, \pi/3)$ and $\xi \equiv 0$ on $(5\pi/3, 2\pi]$. We define the functions

$$w_1(x) := (2\pi - x)^{1/3} \xi(2\pi - x); \quad w_2(x) := x^{2/3} \xi(x); \quad w_3(x) := (2\pi - x)^{2/3} \xi(2\pi - x)$$

and the extensions Λ, Λ_1 of Λ_0 with

$$D(\Lambda) = D(\Lambda_0) + \langle w_1 \rangle + \langle w_2 \rangle; \quad D(\Lambda_1) = D(\Lambda_0) + \langle w_2 \rangle + \langle w_3 \rangle.$$

By Theorems 2 and 4, §18 of [7], Λ and Λ_1 are self-adjoint extensions of Λ_0 . Note that Λ_1 corresponds to Dirichlet boundary conditions for the initial PDE, whereas Λ corresponds to the Dirichlet boundary condition at 0 and the Neumann boundary condition at 2π .

We now prove that Λ_1 has discrete spectrum. It is sufficient to show that for any $A \in \mathbb{R}$, any subspace $L \subset D(\Lambda_1)$ such that all $g \in L$ obey the inequality

$$A(g, g) < (\Lambda_1 g; g) \tag{10}$$

is finitely dimensional. The potential $V(\theta)$ has singularities at 0, 2π of order $\frac{2}{9}\theta^{-2}$ and $\frac{2}{9}(2\pi - \theta)^{-2}$ correspondingly. By Hardy’s inequality, for any $\alpha > \frac{8}{9}$ there exists a constant B such that

$$(Vg, g) \leq \alpha(g', g') + B(g, g), \tag{11}$$

for any $g \in W_2^1((0, 2\pi))$ such that $g(0) = g(2\pi) = 0$, and hence for any $g \in D(\Lambda_1)$. Therefore (10) implies

$$\frac{A - B}{1 - \alpha}(g, g) \leq -(g', g').$$

Since the spectrum of $g \mapsto g''$ defined on $g \in W_2^1((0, 2\pi))$, $g(0) = g(2\pi) = 0$ is discrete and negative, this cone can only contain finitely dimensional subspaces.

By Theorem 2, §19 of [7], this implies that the spectrum of Λ is also discrete. Standard arguments now imply that all eigenvalues of Λ are simple, and that the positive eigenfunction $\psi_0 := (\sin(\theta/2) \sin(\theta/4))^{1/3}$ of Λ corresponds to the largest eigenvalue $-5/144$. It has been shown in [6] that for $\varepsilon = 1$ and any $s > 0$,

$$h^\varepsilon(\theta, s) := \int_0^\varepsilon h(\theta, s + s') ds'$$

satisfies the Neumann boundary condition ($\partial_\theta h^\varepsilon(2\pi, s) = 0$) at 2π and the Dirichlet boundary conditions at 0. First, we observe that the same is true (and with the same proof) for any $\varepsilon > 0$. Second, the functions h^ε are monotone decreasing in s and increasing in θ . Hence (8) implies that $h^\varepsilon(\cdot, s)$ is concave for any s . By some elementary calculus, one deduces from this and (7) that $\partial_\theta h^\varepsilon \leq C\theta^{-2/3}$ as $\theta \rightarrow 0$. Observe that also $\lim_{\theta \rightarrow 2\pi} \partial_\theta h = 0$.

We now consider $f^\varepsilon(\theta, s) = (\sin(\theta/2))^{1/3} h^\varepsilon(\theta, s)$ and write

$$f^\varepsilon(\cdot, s) = \sum c_k(s) \psi_k(\cdot),$$

where ψ_k are eigenfunctions of Λ and the equality holds in the sense of L_2 . By multiplying (9) by ψ_k and integrating by parts, taking into account the boundary conditions for h derived above, we obtain

$$\partial_t c_k(s) = \lambda_k c_k(s),$$

and hence

$$f^\varepsilon(\cdot, s) e^{5/48s} \xrightarrow{L_2} C_\varepsilon \psi_0 \quad \text{as } s \rightarrow \infty,$$

Since the functions $h_\varepsilon(\cdot, s)$ are concave for all s , this actually implies uniform convergence. The corresponding result for h is obtained by taking $\varepsilon \rightarrow 0$, and the constant K_4 is computed by projecting the initial conditions $h(\cdot, 0) \equiv 1$ to the main eigenspace. \square

3. Five-Point Function

To compute F we have to consider the law of SLE(6) curve started from u_2 and conditioned on the event \mathcal{E}_s that $r(u_3) < e^{-s}$. Recall that in terms of interfaces, the event \mathcal{E}_s means that the curve hits $(u_3 - e^{-s}; u_3 + e^{-s})$ before swallowing u_1 , and $\mathbb{P}[\mathcal{E}_s] = C(u_1, u_2, u_3, s)$. Let $u_1(t), u_2(t), u_3(t)$ and $r(u_3)$ be defined on a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, $u_2(t) = B_{6t}$ and $u_{1,3}(t) = g_t(u_{1,3})$, where B is a standard Brownian motion under \mathbb{P} started from u_2 and g_t is the Loewner map driven by $u_2(t)$. Let $\tilde{\mathbb{P}}$ be the measure \mathbb{P} conditioned on \mathcal{E}_s . We study the law of the driving force with respect to $\tilde{\mathbb{P}}$. We have

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_t = \frac{C(u_1(t), u_2(t), u_3(t), s(t))\mathbf{1}_{t \leq T(u_3)} + \mathbf{1}_{t > T(u_3), r(u_3) < e^{-s}}}{C(u_1, u_2, u_3, s)} = D_t,$$

where $T(u_3)$ is the swallowing time for u_3 . By Girsanov’s theorem, \tilde{B}_t is a standard Brownian motion under $\tilde{\mathbb{P}}$ if

$$d\tilde{B}_t = dB_t - \frac{\langle B_t, D_t \rangle}{D_t}.$$

Standard argument now shows that D_t is a martingale, and

$$dD_t = \frac{\partial_{u_2} C(u_1(t), u_2(t), u_3(t), s(t))}{C(u_1(t), u_2(t), u_3(t), s(t))} \sqrt{6} dB_t,$$

which implies that

$$dB_t = d\tilde{B}_t + \sqrt{6} \frac{\partial_{u_2} C(u_1(t), u_2(t), u_3(t), s(t))}{C(u_1(t), u_2(t), u_3(t), s(t))} dt.$$

This proves that under condition $r(u_3) < e^{-s}$ the driving force of the Loewner evolution becomes

$$du_2(t) = \sqrt{6} d\tilde{B}_t + 6 \frac{\partial_{u_2} C(u_1(t), u_2(t), u_3(t), s(t))}{C(u_1(t), u_2(t), u_3(t), s(t))} dt.$$

The direct computation shows that in the limit, as $s \rightarrow \infty$, one has

$$du_2 = \sqrt{6} \tilde{B}_t + \left(\frac{-2}{u_2 - u_3} + \frac{2}{u_2 - u_1} \right) dt,$$

which is the driving force of SLE(6, 2, -2) (see [1], where these processes were introduced) started from u_2, u_1 and u_3 . We would like to point out that the force point with $\rho = 2 = 6 - 4$ conditions SLE(6) not to swallow the force point and $\rho = -2 = 6 - 8$ conditions SLE(6) to hit the force point.

Remark 3. For general κ , the above argument does not work: \mathcal{E}_s is now a five-point event and its probability is not expressed in terms of Cardy’s four-point formula. By conditioning first on hitting u_3 , and then on not swallowing u_1 , it is still possible to compute the drift in the limit $s \rightarrow \infty$; this leads to

$$du_2 = \sqrt{\kappa} \tilde{B}_t + \left(\frac{\kappa - 8}{u_2 - u_3} + \frac{\kappa G'(\frac{u_2 - u_1}{u_3 - u_1})}{(u_3 - u_1) G(\frac{u_2 - u_1}{u_3 - u_1})} \right) dt,$$

where $G(x) = \int_0^x \theta^{-\frac{4}{\kappa}} (1 - \theta)^{\frac{2(6-\kappa)}{\kappa}} d\theta$. Hence for $\kappa \neq 6$ conditioning on both events is not the same as adding corresponding force points.

Let τ be the stopping time which is the minimum of $T(w)$ – the swallowing time for w and $T(u_3)$ – the swallowing time for u_3 . By Ω we denote the component of the complement of the SLE trace up to time τ which contains the point w .

For a particular realization $\gamma([0, \tau])$ of conditioned SLE interface, the probability F is the probability that the conformal radius of Ω in the complement of the cluster which is attached to the union of the left side of γ and $[u_1, u_2]$ is at most e^{-s} . Let E be the intersection of the left side of γ and $[u_1, u_2]$ with the boundary of Ω , and let ϕ be the conformal map from Ω onto the unit disc which maps w to the origin. By conformal invariance, the probability of our event is the same as the probability that in the unit disc the conformal radius about the origin of the complement of the percolation cluster which is attached to $\phi(E)$ is at most $e^{-s}|\phi'(w)|$. By Lemma 2.2, this probability behaves as

$$K_4 e^{-s \frac{5}{48}} |\phi'(w)|^{5/48} (\sin(\pi\omega/2))^{1/3},$$

where ω is the harmonic measure of E seen from w . This proves that after factoring out $e^{-s5/48}$ and passing to the limit we have

$$F(u_1, u_2, u_3, w) = K_4 \mathbb{E} \left[|\phi'(w)|^{5/48} (\sin(\pi\omega/2))^{1/3} \right],$$

where the expectation is taken with respect to the law of $\gamma([0, \tau])$.

Lemma 3.1. *The function F is a solution of the following PDE:*

$$\begin{aligned} \Delta_F F &= \frac{5}{96} \frac{-2}{(w - u_2)^2} F + \frac{5}{96} \frac{-2}{(\bar{w} - u_2)^2} F + \frac{2}{u_1 - u_2} \partial_{u_1} F + \frac{2}{u_3 - u_2} \partial_{u_3} F \\ &+ 3\partial_{u_2}^2 F + 6 \frac{\partial_{u_2} C}{C} \partial_{u_2} F + \frac{2}{w - u_2} \partial_w F + \frac{2}{\bar{w} - u_2} \partial_{\bar{w}} F = 0. \end{aligned} \tag{12}$$

Proof. As we proved before, γ has a law of SLE(6; -2, 2) trace. The driving function of this SLE is given by the solution of the SDEs

$$\begin{aligned} du_2(t) &= \sqrt{6} d\tilde{B}_t + \left(\frac{-2}{u_2(t) - u_3(t)} + \frac{2}{u_2(t) - u_1(t)} \right) dt = \sqrt{6} d\tilde{B}_t + 6 \frac{\partial_{u_2} C}{C} dt, \\ du_1(t) &= \frac{2dt}{u_1(t) - u_2(t)}, \\ du_3(t) &= \frac{2dt}{u_3(t) - u_2(t)}. \end{aligned}$$

The domain Markov property implies that

$$|g'_t(w)|^{5/48} F(u_1(t), u_2(t), u_3(t), w(t))$$

is equal to $K_4 \mathbb{E}[|\phi'(w)|^{5/48} (\sin(\pi\omega/2))^{1/3} | \mathcal{F}_t]$ and hence is a martingale (here $w(t) = g_t(w)$ is the image of w under the SLE map g_t). It is more convenient to write Itô's formula with respect to w and \bar{w} instead of $\text{Re } w$ and $\text{Im } w$. In these terms

$$g'_t(w)^{5/96} g'_t(\bar{w})^{5/96} F(u_1(t), u_2(t), u_3(t), w(t), \bar{w}(t))$$

is a martingale. By Itô's formula the drift is equal to zero, which is equivalent to Eq. (12).

Theorem 3.2. *The product of two functions F and C is annihilated by the following operator:*

$$\Lambda = -\frac{5}{48} \frac{1}{(w - u_2)^2} - \frac{5}{48} \frac{1}{(\bar{w} - u_2)^2} - \frac{2}{3} \frac{1}{u_3 - u_2} + \frac{2}{u_1 - u_2} \partial_{u_1} + \frac{2}{u_3 - u_2} \partial_{u_3} + 3\partial u_2^2 + \frac{2}{w - u_2} \partial_w + \frac{2}{\bar{w} - u_2} \partial_{\bar{w}}. \tag{13}$$

Proof. If we apply Λ to FC then the result will be

$$(\Lambda F) C + F \left(\frac{2}{u_1 - u_2} \partial_{u_1} C + \frac{2}{u_3 - u_2} \partial_{u_3} C + 3\partial u_2^2 C \right) + 6\partial_{u_2} F \partial_{u_2} C.$$

Using that

$$\Lambda = \Lambda_F - 6 \frac{\partial_{u_2} C}{C} \partial_{u_2} - \frac{2}{3} \frac{1}{u_3 - u_2},$$

the formula above can be rewritten as

$$F \left(-\frac{2}{3} \frac{1}{u_3 - u_2} C + \frac{2}{u_1 - u_2} \partial_{u_1} C + \frac{2}{u_3 - u_2} \partial_{u_3} C + 3\partial u_2^2 C \right),$$

which is equal to zero by (5). \square

Equation (13) is exactly Eq. (A.5) from [11]. Following [11] we can transfer this equation into the semi-infinite strip (keeping covariance in mind) where after the right ansatz one can separate variables and find a solution to Eq. (13):

$$G(x, y) := \sinh(\pi x)^{-1/3} e^{\pi x/3} {}_2F_1 \left(-\frac{1}{2}, -\frac{1}{3}, \frac{7}{6}, e^{-2\pi x} \right) \times \left(\frac{\sinh(\pi x)^2 \sin(\pi y)^2}{\sinh(\pi x)^2 + \sin(\pi y)^2} \right)^{11/96}, \tag{14}$$

where x and y are real and imaginary parts of the image of w under the conformal transformation which maps the half-plane with marked point u_1, u_2 and u_3 onto the semi infinite strip $\mathbb{S} = \{0 < x < \infty, 0 < y < 1\}$.

This solution to (13) is not unique. In order to prove that G indeed gives the correct formula for the right-hand side of (4), we proceed as follows. Let ψ_{u_1, u_2, u_3} (or ψ for short) be the conformal map that transforms $\mathbb{H}, u_1, u_2, u_3$ to $\mathbb{S}, i, 0, \infty$. If we repeat the above Itô computations with

$$H(u_1, u_2, u_3, w) := K_5 |\psi'_{u_1, u_2, u_3}(w)|^{5/48} G(\text{Re } \psi_{u_1, u_2, u_3}(w), \text{Im } \psi_{u_1, u_2, u_3}(w))$$

instead of F , we find that

$$|g'_t(w)|^{5/48} H(u_1(t), u_2(t), u_3(t), w(t))$$

is a local martingale for the $SLE(6, 2, -2)$ process. We should now check that (for a right choice of the constant K_5) it has a correct value at the stopping time τ , that is

$$\lim_{t \rightarrow \tau} |g'_t(w)|^{5/48} F(u_1(t), u_2(t), u_3(t), w(t)) = \lim_{t \rightarrow \tau} |g'_t(w)|^{5/48} H(u_1(t), u_2(t), u_3(t), w(t)). \tag{15}$$

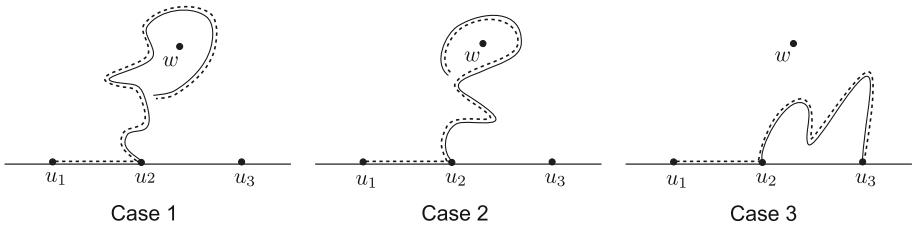


Fig. 1. Three possible stopping scenarios for the SLE(6, 2, -2) interface. Case 1: at $t = \tau$ the curve cuts the point w out of the cluster K connected to (u_1, u_2) . Case 2: at $t = \tau$, the whole boundary, as seen from w , belongs to K . Case 3: only the dashed piece of the boundary is explored to be a part of K

Since H is an explicit smooth function, all the Itô computations are justified. We will finish the proof by checking uniform integrability and applying the optional stopping theorem to $|g'_t(w)|^{5/48}(H - F)$ to conclude that it is identically zero.

There are three possible stopping scenarios (see Fig. 1). Consider them separately. In all cases ϕ_t denotes a conformal map from $\mathbb{H} \setminus \gamma_t$ to the unit disc with $\phi_t(w) = 0$.

Case 1. The curve swallows w before hitting u_3 (i.e. $\tau = T(w)$) and does that by closing a clockwise loop. In this case the left-hand side of (15) tends to zero. However, the limit $\lim_{t \rightarrow \tau} |\phi'_t(w)|$ exists and is finite almost surely. We can write $\phi_t =: h_t \circ g_t$, where $h_t(z) = \frac{w_t - z}{z - \bar{w}_t}$. It is hence sufficient to show that

$$\begin{aligned} & |h'_t(w(t))|^{-5/48} H(u_1(t), u_2(t), u_3(t), w(t)) \\ &= (2\text{Im } w_t)^{5/48} H(u_1(t), u_2(t), u_3(t), w(t)) \end{aligned}$$

tends to zero as $w(t)$ approaches u_2 in such a way that the harmonic measure of (u_1, u_2) seen from $w(t)$ tends to zero. Without loss of generality, we can assume that $u_1, u_2, u_3 = 0, 1, \infty$. Then

$$(\psi_1)^{-1} := (\psi_{0,1,\infty})^{-1} = \frac{\cosh(\pi z) + 1}{2}, \tag{16}$$

and we have the following expansion around 1:

$$|\psi'_1(z)| = \frac{|z - 1|^{-1/2}}{\pi} + o(|z - 1|^{-1/2}), \quad |z| \rightarrow 1. \tag{17}$$

It remains to plug everything into the definition of H , taking into account that $|\text{Re } \psi(w(t))| \gg |\text{Im } \psi(w(t))|$, and we are done.

Case 2. The curve swallows w before hitting u_3 (i.e. $\tau = T(w)$) and does that by closing a counterclockwise loop. In this case, the left-hand side of (15) tends to $K_4 |\phi'(w)|^{5/48}$. Hence, it is sufficient to prove that

$$F(u_1(t), u_2(t), u_3(t), w(t)) \sim H(u_1(t), u_2(t), u_3(t), w(t))$$

as $w(t)$ approaches u_2 in such a way that the harmonic measure of (u_1, u_2) seen from $w(t)$ tends to 1 (that is, $|\text{Re } \psi(w(t))| \ll |\text{Im } \psi(w(t))|$). One has

$$F(u_1(t), u_2(t), u_3(t), w(t)) \sim K_4 |h'_t(w(t))|^{5/48} \sim K_4 (2\text{Im } w_t)^{-5/48}.$$

Once again, we can assume without loss of generality that $u_1, u_2, u_3 = 0, 1, \infty$. Then, denoting $x(t) = \operatorname{Re} \psi(w(t))$, $y(t) = \operatorname{Im} \psi(w(t))$, one has

$$G(x(t), y(t)) \sim {}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}, \frac{7}{6}, 1\right) (\pi x(t))^{-5/48}$$

and

$$|\psi'(w(t))| \sim \frac{|w(t) - 1|^{-1/2}}{\pi} \sim \frac{2}{\pi^2} |y(t)|^{-1}.$$

Hence

$$\begin{aligned} H(u_1(t), u_2(t), u_3(t), w(t)) &\sim K_5 {}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}, \frac{7}{6}, 1\right) \left(\frac{2}{\pi^3 x(t)y(t)}\right)^{5/48} \\ &\sim K_5 {}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}, \frac{7}{6}, 1\right) \left(\frac{2}{\pi}\right)^{5/48} \frac{1}{(2\operatorname{Im} w(t))^{5/48}}. \end{aligned}$$

Case 3. The curve hits u_3 before swallowing w (i.e. $\tau = T(u_3)$). In this case, we should compare the limits of H and F when $u_3 \rightarrow u_2$. For F , the answer is given by Lemma 2.2

$$\begin{aligned} \lim_{u_3 \rightarrow u_2} F(u_1, u_2, u_3, w) \\ = K_4 |\phi'(w)|^{5/48} (\sin(\pi\omega/2))^{1/3} = \frac{K_4}{(2\operatorname{Im} w)^{5/48}} (\sin(\pi\omega/2))^{1/3}, \end{aligned} \tag{18}$$

where ϕ is a conformal map from the upper half-plane to the unit disc that maps w to the origin, and ω is the harmonic measure of (u_1, u_2) seen from w .

Consider now the limit of H . Note that as $u_3 \rightarrow u_2$, the image $\psi(w)$ tends to i along some direction. We write $\psi = \psi_1 \circ \psi_2$, where ψ_2 maps $\mathbb{H}, u_1, u_2, u_3$ to $\mathbb{H}, 0, 1, \infty$ and ψ_1 maps $\mathbb{H}, 0, 1, \infty$ to $\mathbb{S}, i, 0, \infty$. It is convenient to choose $r = |\psi_2(w)|$ as a small parameter. Taking (16) and (17) into account, one gets, as $u_3 \rightarrow u_2$:

$$\begin{aligned} \psi_2(w) &= \frac{w - u_1}{w - u_3} \frac{u_2 - u_3}{u_2 - u_1} =: r e^{i\theta}, \\ |\psi_2'(w)| &= r \frac{|u_1 - u_3|}{|w - u_1||w - u_3|}, \\ x := \operatorname{Re} \psi(w) &= \frac{2}{\pi} r^{\frac{1}{2}} \sin \frac{\theta}{2} + o(r^{\frac{1}{2}}), \\ y := \operatorname{Im} \psi(w) &= 1 - \frac{2}{\pi} r^{\frac{1}{2}} \cos \frac{\theta}{2} + o(r^{\frac{1}{2}}), \\ |\psi_1'(\psi_2(w))| &= \frac{r^{-\frac{1}{2}}}{\pi} + o(r^{-\frac{1}{2}}). \end{aligned}$$

Plugging everything into the definition of H , we find that

$$\begin{aligned} \lim_{u_3 \rightarrow u_2} H &= K_5 \frac{2^{-1/3}}{\pi^{5/48}} {}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}, \frac{7}{6}, 1\right) \sin(\theta/2)^{-\frac{1}{3}} \sin(\theta)^{11/48} \\ &\times \left(\frac{|u_1 - u_2|}{|w - u_1||w - u_2|}\right)^{5/48}. \end{aligned}$$

Note that $1 - \theta/\pi$ is actually equal to the harmonic measure of u_1, u_2 seen from w . Simple computation now shows that

$$\frac{|u_1 - u_2|}{|w - u_1||w - u_2|} = \frac{\sin(\theta)}{\text{Im } w},$$

hence

$$\lim_{u_3 \rightarrow u_2} H = K_5 \frac{2^{5/48}}{\pi^{5/48}} {}_2F_1 \left(-\frac{1}{2}, -\frac{1}{3}, \frac{7}{6}, 1 \right) \frac{(\cos \theta/2)^{1/3}}{(2\text{Im } w)^{5/48}}.$$

Combining the results of all three cases, we see that Eq. (15) is satisfied if we choose

$$K_5^{-1} = \frac{2^{5/48}}{K_4 \pi^{5/48}} {}_2F_1 \left(-\frac{1}{2}, -\frac{1}{3}, \frac{7}{6}, 1 \right). \tag{19}$$

The only remaining part is applicability of the optional stopping theorem to the local martingale $|g'_t(w)|^{5/48}(F - H)$. Note first of all that there exists a constant M such that

$$|g'_t(w)|^{5/48} F \leq M |\phi'_t(w)|^{5/48},$$

since the right-hand side is proportional to the probability that w is connected to the boundary of the domain. A direct computation shows the same bound holds for H :

$$|g'_t(w)|^{5/48} H \leq M |\phi'_t(w)|^{5/48}.$$

Hence it is sufficient to prove that $|\phi'_t(w)|^{5/48}$ is uniformly integrable. Since this is monotone increasing in t , it is sufficient to show that $\mathbb{E}|\phi'_t(w)|^{5/48} < \infty$. This bound follows from known estimates on the dimension of SLE_6 (see e. g. [8], Lemma 6.3), concluding the proof of Theorem 1.1.

4. A Factorization Formula

In this section we prove the factorization formula obtained by Simmons, Kleban and Ziff in [10]. We introduce the following correlation functions:

$$\begin{aligned} P_2(u_1, u_3) &:= \lim_{u_2 \rightarrow u_1} C(u_1, u_2, u_3) \left| \frac{u_2 - u_1}{2} \right|^{-1/3}, \\ F(u_1, u_2, w) &:= \lim_{r \rightarrow \infty} F(u_1, u_2, w, r) e^{5/48r}, \\ P_3(u_1, w) &:= \lim_{u_2 \rightarrow u_1} F(u_1, u_2, w) \left| \frac{u_2 - u_1}{2} \right|^{-1/3}, \\ P_4(u_1, u_3, w) &:= \lim_{u_2 \rightarrow u_1} C(u_1, u_2, u_3) F(u_1, u_2, u_3, w) \left| \frac{u_2 - u_1}{2} \right|^{-1/3}. \end{aligned}$$

Proposition 4.1. *One has the following factorization formula:*

$$P_4^2(u_1, u_3, w) = K_F P_3(u_1, w) P_3(u_2, w) P_2(u_1, u_3), \tag{20}$$

where

$$K_F = \frac{K_5^2 K_3 2^{25/24}}{K_4^2 \pi^{5/24}} = \frac{2\sqrt{\pi}}{2F_1^2 \left(-\frac{1}{2}, -\frac{1}{3}, \frac{7}{6}, 1 \right) \Gamma(\frac{1}{3}) \Gamma(\frac{7}{6})} = \frac{2^7 \pi^5}{3^{3/2} \Gamma(\frac{1}{3})^9}.$$

Proof. The proposition is proven by direct computation. One gets immediately from Lemma 2.1 that

$$P_2(u_1, u_3) = K_3 \frac{2^{1/3}}{(u_1 - u_3)^{2/3}}. \tag{21}$$

In the notatin of Lemma 2.2, one has

$$\pi\omega \sim \frac{|u_1 - u_2| \operatorname{Im} w}{|w - u_1|^2}, \quad u_2 \rightarrow u_1,$$

and $|\phi'(w)| = \frac{1}{2\operatorname{Im} w}$, hence

$$P_3(u_1, w) = \frac{K_4}{2^{5/48}} \frac{(\operatorname{Im} w)^{11/48}}{|u_1 - w|^{2/3}}. \tag{22}$$

It remains to figure out the limit of $F(u_1, u_2, u_3, w)$ as $u_2 \rightarrow u_1$. Note that in this case the image $\psi_{u_1, u_2, u_3}(w)$ of w under the mapping to the strip tends to infinity. We can write, as before, $\psi_{u_1, u_2, u_3} = \psi_1 \circ \psi_2$, where ψ_2 maps \mathbb{H} , u_1, u_2, u_3 to \mathbb{H} , $0, 1, \infty$ and

$$\psi_1^{-1}(z) = \frac{\cosh(\pi z) + 1}{2}.$$

Then we have

$$\begin{aligned} \psi_2(w) &\sim -\frac{(w - u_1)(u_3 - u_1)}{(w - u_3)|u_2 - u_1|}, \\ \psi_2'(w) &\sim \frac{(u_3 - u_1)^2}{(w - u_3)^2|u_1 - u_2|}. \end{aligned}$$

Writing $x = \operatorname{Re} \psi_{u_1, u_2, u_3}(w)$ and $y = \operatorname{Im} \psi_{u_1, u_2, u_3}(w)$, we obtain

$$2\psi_2(w) \sim e^{\pi x} \frac{\cos(\pi y) + i \sin(\pi y)}{2},$$

hence

$$\begin{aligned} e^{\pi x} &\sim |4\psi_2(w)| \sim \frac{4|w - u_1|(u_3 - u_1)}{|w - u_3||u_2 - u_1|}, \\ \pi y &\sim \arg \psi_2(w) = \arg \frac{-w + u_1}{w - u_3} = \pi - \zeta, \\ \psi_1'(\psi_2(w)) &\sim \frac{4}{\pi e^{\pi x}} \sim \frac{|w - u_3||u_2 - u_1|}{\pi |w - u_1||u_3 - u_1|}, \end{aligned}$$

where ζ is the angle at w in the triangle (u_1, u_3, w) . Plugging everything into (4), we get that

$$\lim_{u_2 \rightarrow u_1} F(u_1, u_2, u_3, w) = \frac{K_5 2^{1/3}}{\pi^{5/48}} (\sin \zeta)^{1/3} \operatorname{Im} w^{-5/48}. \tag{23}$$

The proposition now follows from (21), (22) and (23). \square

Remark 4. The proposition has the following probabilistic interpretation. Let $P_2^\varepsilon(u_1, u_2)$, $P_3^\varepsilon(u, w)$ and $P_4^\varepsilon(u_1, u_2, w)$ be the probabilities that ε -neighborhoods of corresponding points are connected by a percolation cluster (we understand ε -neighborhoods in the sense of $r(u_i)$ and $r(w)$, as defined in the Introduction). Then (20) holds with P_i replaced by P_i^ε and equality replaced by equivalence as $\varepsilon \rightarrow 0$. Indeed, by definition $P_i^\varepsilon \sim P_i \varepsilon^{\sigma_i}$ with an appropriate σ_i . It is immediate to see that these power factors cancel out once plugged into (20).

Remark 5. The constants K_3, K_4, K_5 in this paper are non-universal, in particular, they depend on our definition of neighborhoods of points (and in the lattice formulation they would depend on the lattice). The constant K_F , however, is conjectured to be universal, as all non-universal factors cancel out.

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