Sharp estimates of integral functionals on classes of functions with small mean oscillation

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Abstract

We unify several Bellman function problems treated in [1, 2, 4, 5, 6, 9, 10, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24] into one setting. For that purpose we define a class of functions that have, in a sense, small mean oscillation (this class depends on two convex sets in $\mathbb{R}^2$). We show how the unit ball in the BMO space, or a Muckenhoupt class, or a Gehring class can be described in such a fashion. Finally, we consider a Bellman function problem on these classes, discuss its solution and related questions.

Since Slavin [12] and Vasyunin [18] proved the sharp form of the John–Nirenberg inequality (see [15]), there have been many papers where similar principles are used to prove sharp estimates of this kind. However, there is no theory or even a unifying approach; moreover, the class of problems to which the method can be applied has not been described yet. There is a portion of heuristics in the folklore that is each time applied to a new problem in a very similar manner. The first attempt to build a theory (at least for BMO) was made in [16], then the theory was developed in the paper [4] (see the short report [5] also). We would also like to draw the reader's attention to the forthcoming paper [6], which can be considered as a description of the theory for the BMO space in a sufficient generality. Problems of this kind were considered not only in BMO, but in Muckenhoupt classes, Gehring classes, etc (see [1, 2, 11, 13, 19, 20]). In this short note, we define a class of functions and an extremal problem on it that includes all the problems discussed above. We believe that the unification we offer gives a strong basis for a theory that will distinguish a certain class of problems to which the method is applicable in a direct way. In Section 1 we state the problem and discuss related questions. Section 2 contains a detailed explanation how our classes of functions include the unit ball in BMO as well as the "unit balls" in Muckenhoupt classes and Gehring classes. Finally, in Section 3 we give hints to the solution of the problem (as the reader may expect looking at previous papers, it is rather lengthy and technical, so we omit a description of the solution, but concentrate on an analogy with the case of BMO considered in [4, 6, 22]).

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1 Setting

Let Ω_0 be a non-empty open strictly convex subset of $\mathbb{R}^2$ and let Ω_1 be open strictly convex subset of Ω_0. We define the domain Ω as $\mathrm{cl}(\Omega_0 \setminus \Omega_1)$ (the word "domain" comes from "domain of a function"; the symbol cl denotes the closure) and the class $\boldsymbol{A}_\Omega$ of summable $\mathbb{R}^2$-valued functions on an interval $I \subset \mathbb{R}$ as follows:

$$\boldsymbol{A}_\Omega = \big\{\varphi \in L^1(I,\mathbb{R}^2) \;\mid\; \varphi(I) \subset \partial\Omega_0 \quad \text{and} \quad \forall \text{ subinterval } J \subset I \quad \langle\varphi\rangle_J \notin \Omega_1\big\}. \tag{1.1}$$

Here $\langle\varphi\rangle_J = \frac{1}{|J|}\int_J \varphi(s)\,ds$ is the average of φ over J. In Section 2 we show how the unit ball in BMO as well as the "unit balls" in Muckenhoupt and Gehring classes can be represented in the form (1.1). Let $f\colon \partial\Omega_0 \to \mathbb{R}$ be a bounded from below Borel measurable locally bounded function. We are interested in sharp bounds for the expressions of the form $\langle f(\varphi)\rangle_I$, where $\varphi \in \boldsymbol{A}_\Omega$.

Again, in Section 2 we explain how the John–Nirenberg inequality or other inequalities of harmonic analysis can be rewritten as estimations of such an expression. The said estimates are delivered by the corresponding Bellman function

$$\boldsymbol{B}_{\Omega,f}(x) = \sup\big\{\langle f(\varphi)\rangle_I \;\big|\; \langle\varphi\rangle_I = x,\ \varphi \in \boldsymbol{A}_\Omega\big\}. \tag{1.2}$$

Problem 1.1. *Given a domain Ω and a function f, calculate the function $\boldsymbol{B}_{\Omega,f}$.*

As it has been said in the abstract, the particular cases of this problem were treated in the papers [1, 2, 4, 5, 6, 9, 10, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24] (see Section 2 for a detailed explanation). The main reason for Problem 1.1 to be solvable (and it has been heavily used in all the preceeding work) is that the function $\boldsymbol{B}$ enjoys good properties.

Definition 1.2. *Let ω be a subset of $\mathbb{R}^d$. We call a function $G\colon w \to \mathbb{R}\cup\{+\infty\}$ locally concave on ω if for every segment $\ell \subset \omega$ the restriction $G\big|_\ell$ is concave.*

Define the class of functions on Ω:

$$\Lambda_{\Omega,f} = \Big\{G\colon \Omega \to \mathbb{R}\cup\{+\infty\} \;\Big|\; G \text{ is locally concave on } \Omega, \quad \forall x \in \partial\Omega_0 \quad G(x) \geqslant f(x)\Big\}. \tag{1.3}$$

The function $\mathfrak{B}_{\Omega,f}$ is given as follows:

$$\mathfrak{B}_{\Omega,f}(x) = \inf_{G\in\Lambda_{\Omega,f}} G(x). \tag{1.4}$$

Conjecture 1.3. $\boldsymbol{B}_{\Omega,f} = \mathfrak{B}_{\Omega,f}$.

In particular, the conjecture states that the Bellman function is locally concave (because the function $\mathfrak{B}_{\Omega,f}$ is).

Problem 1.4. *Prove Conjecture* 1.3 *in adequate generality.*

Though it may seem that one should solve Problem 1.4 before turning to Problem 1.1, it is not really the case. All the preceeding papers used Conjecture 1.3 as an assumption that allowed the authors to guess $\boldsymbol{B}$, then prove that this function was the Bellman function indeed, and only then verify Conjecture 1.3 for Ω and f chosen. However, to treat Problem 1.4 in itself, one has to invent a different approach, see Section 3.

We note that one should impose some additional conditions on Ω and f to provide a solution to the problems. We postpone the detailed discussion of this to Section 3 and pass to examples.

2 Examples

From now on, we follow the agreement: if $g\colon \mathbb{R} \to \mathbb{R}^2$ is some fixed parametrization of $\partial\Omega_0$, then the function $f(g)\colon \mathbb{R} \to \mathbb{R}$ is denoted by $\tilde{f}$.

The BMO **space.** We consider the BMO space with the quadratic seminorm. Let ε be a positive number. Set $\Omega_0 = \{x \in \mathbb{R}^2 \mid x_1^2 < x_2\}$ and $\Omega_1 = \{x \in \mathbb{R}^2 \mid x_1^2 + \varepsilon^2 < x_2\}$. A function

$$\varphi = (\varphi_1, \varphi_2)\colon I \to \partial\Omega_0$$

belongs to the class $\boldsymbol{A}_\Omega$ if and only if its first coordinate φ_1 belongs to BMO_ε (the ball of radius ε in BMO). Indeed, for any $t \in I$ we have $\varphi_2(t) = \varphi_1^2(t)$, therefore, the condition $\langle\varphi\rangle_{J} \notin \Omega_1$ can be rewritten as

$$\langle\varphi_1^2\rangle_{J} \leqslant \langle\varphi_1\rangle_{J}^2 + \varepsilon^2,$$

which is the same as

$$\big\langle\big(\varphi_1 - \langle\varphi_1\rangle_{J}\big)^2\big\rangle_{J} \leqslant \varepsilon^2. \tag{2.1}$$

Now we see that the class $\boldsymbol{A}_\Omega$ corresponds to BMO_ε. The Bellman function (1.2) estimates the functional $\langle\tilde{f}(\varphi_1)\rangle_{I}$. The solution of Problem 1.1 with $\tilde{f}(t) = e^{\lambda t}$ leads to the John–Nirenberg inequality in its integral form, the case $\tilde{f}(t) = \chi_{(-\infty,-\lambda]\cup[\lambda,\infty)}(t)$ corresponds to the weak form of the John–Nirenberg inequality, and the case $f(t) = |t|^p$ leads to equivalent defintions of BMO. We address the reader to the paper [4] for a detailed discussion. This case is the subject of study for the papers [4, 6, 9, 10, 15, 16, 21, 22].

Classes A_{p_1,p_2}**.** Let p_1 and p_2, $p_1 > p_2$, be real numbers and let $Q \geqslant 1$. Suppose

$$\Omega_0 = \{x \in \mathbb{R}^2 \mid x_1, x_2 > 0,\ x_2^{\frac{1}{p_2}} < x_1^{\frac{1}{p_1}}\} \quad \text{and} \quad \Omega_1 = \{x \in \mathbb{R}^2 \mid x_1, x_2 > 0,\ Qx_2^{\frac{1}{p_2}} < x_1^{\frac{1}{p_1}}\}.$$

If a function φ belongs to the class $\boldsymbol{A}_\Omega$, then its first coordinate φ_1 belongs to the so-called A_{p_1,p_2} class. The "norm" in this class is defined as

$$[\psi]_{A_{p_1,p_2}} = \sup_{J\subset I}\ \langle\psi^{p_1}\rangle_{J}^{\frac{1}{p_1}}\langle\psi^{p_2}\rangle_{J}^{-\frac{1}{p_2}}, \tag{2.2}$$

where the supremum is taken over all subintervals of I. These classes were introduced in [20]. If $p \in (1,\infty)$, then $A_{1,-\frac{1}{p-1}} = A_p$, where A_p stands for the classical Muckenhoupt class. The limiting cases A_1 and A_∞ also fit into this definition (with Hruschev's "norm" on A_∞). When $p_2 = 1$ and $p_1 > 1$, the class A_{p_1,p_2} coincides with the so-called Gehring class (see [7] or [8]). One can see that the functions in the Gehring class are exactly those that satisfy the reverse Hölder inequality. Sometimes, the Gehring class is called the reverse-Hölder class. Estimates of integral functionals as provided by the Bellman function (1.2) lead to various sharp forms of the reverse Hölder inequality, see [20]. These cases were treated in the papers [1, 2, 11, 19, 20].

Reverse Jensen classes. These classes were introduced in [7]. Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function. Let $Q > 1$. Consider the class of functions $\psi\colon I \to \mathbb{R}_+$ such that

$$\forall J \subset I \quad \langle\Phi(\psi)\rangle_{J} \leqslant Q\Phi(\langle\psi\rangle_{J}).$$

Surely, both a Muckenhoupt class and a Gehring class can be described as certain Reverse Jensen classes. The corresponding domain is $\{x \in \mathbb{R}^2 \mid x_1, x_2 \geqslant 0,\ \Phi(x_1) \leqslant x_2 \leqslant Q\Phi(x_1)\}$. Consult a very recent paper [13], where the Bellman function on the domain $\{x \in \mathbb{R}^2 \mid e^{x_1} \leqslant x_2 \leqslant Ce^{x_1}\}$, $C > 1$, provides sharp constants in the John–Nirenberg inequality for the BMO space equipped with the L^p-type seminorm.

3 Hints to solutions

First, we note that strict convexity of Ω_0 implies the fact that $\boldsymbol{B}(x) = f(x)$ for $x \in \partial\Omega_0$. Second, we need Ω to fulfill several assumptions that all the domains listed in Section 2 do satisfy.

1. The domains Ω_0 and Ω_1 are unbounded. (3.1)
2. The boundary of Ω_1 is C^2-smooth. (3.2)
3. Every ray inside Ω_0 can be translated to belong to Ω_1 entirely. (3.3)

The first two conditions are technical in a sense, the third one is essential, since (under assumption (3.1)) it is equivalent to the fact that for any $x \in \Omega$ there exists a function $\varphi \in \boldsymbol{A}_\Omega$ such that $\langle\varphi\rangle_I = x$ (i.e. the supremum in formula (1.2) is taken over a non-empty set). Now we are ready to present a solution of Problem 1.4.

Theorem 3.1. *Let the domain Ω satisfy the conditions* (3.1), (3.2), (3.3). *If the function f is bounded from below, then $\mathfrak{B}_{\Omega,f} = \boldsymbol{B}_{\Omega,f}$.*

The condition that f is bounded from below is not necessary. However, we note that without this condition the extremal problem in formula (1.2) is not well posed (the integral of $f(\varphi)$ may be not well defined). In [17] the reader can find the proof of Theorem 3.1 for the case $\mathrm{cl}\,\Omega_1 \subset \Omega_0$ as well as its analog where f can be unbounded from below.

To solve Problem 1.1, we need to consider even more restrictive conditions, we introduce some notation for that purpose. Choose $g = (g_1, g_2)\colon \mathbb{R} \to \mathbb{R}^2$ to be a continuous parametrization of $\partial\Omega_0$; let the domain Ω lie on the left of this oriented curve. For any number $u \in \mathbb{R}$ we draw two tangents from the point $g(u)$ to the set Ω_1; by a tangent we mean not a line, but a segment connecting $g(u)$ with the tangency point. We denote the lenghts of the left and the right tangents by $\ell_{\mathrm{L}}(u)$ and $\ell_{\mathrm{R}}(u)$ correspondingly (the left tangent lies between the right one and g', see [4] for explanations about this notation).

1. The boundaries $\partial\Omega_0$ and $\partial\Omega_1$ are C^3-smooth curves, the function f is C^3-smooth. (3.4)
2. The curve $\gamma(t) = \big(g_1(t), g_2(t), \tilde{f}(t)\big) \subset \mathbb{R}^3$ changes the sign of its torsion only a finite number of times. (3.5)
3. The integrals $\int_{-\infty}^{0} \frac{1}{\ell_{\mathrm{R}}}$ and $\int_{0}^{+\infty} \frac{1}{\ell_{\mathrm{L}}}$ diverge. (3.6)

In Condition (3.6) the integration is with respect to the natural parametrization of the curve $\partial\Omega_1$, where the functions ℓ_{R} and ℓ_{L} are considered as the functions of their tangency points lying on $\partial\Omega_1$. For the case where $g(t) = (t, t^2)$ treated in [4], Condition (3.5) turns into "the function $\tilde{f}'''$ changes its sign only a finite number of times"; this is exactly the regularity condition we used in [6]. The last Condition (3.6) is more mysterious, we believe that our considerations may work without it.

We also need a summability assumption for the function f. Let $\alpha_{\mathrm{R}}(u)$ denote the oriented angle between the right tangent at the point u and the vector $(1,0)$, let $\alpha_{\mathrm{L}}(u)$ denote the oriented angle between the left tangent at the point u and the vector $(1,0)$. Then, the summability condition requires the bulky integral

$$\int\limits_{-\infty}^{t} \exp\bigg(\int\limits_{\tau}^{t} \frac{g_1'}{\ell_{\mathrm{R}}\cos(\alpha_{\mathrm{R}})}\bigg) \frac{\tan(\alpha_{\mathrm{R}}(\tau))g_1'(\tau) - g_2'(\tau)}{(g_1'(\tau)g_2''(\tau) - g_2'(\tau)g_1''(\tau))^2} \begin{vmatrix} \tilde{f}'(\tau) & \tilde{f}''(\tau) & \tilde{f}'''(\tau) \\ g_1'(\tau) & g_1''(\tau) & g_1'''(\tau) \\ g_2'(\tau) & g_2''(\tau) & g_2'''(\tau) \end{vmatrix} d\tau \qquad (3.7)$$

to converge for any $t \in \mathbb{R}$ provided γ has negative torsion in a neighborhood of $-\infty$ (and a similar condition with R replaced by L and with $-\infty$ replaced by $+\infty$ provided γ has positive torsion in a neighborhood of ∞).

Claim: under Conditions (3.1)**,** (3.3)**,** (3.4)**,** (3.5)**,** (3.6)**, and the mentioned convergence conditions for the integrals** (3.7) **we can solve Problem 1.1**.

As in [4], by "solution" we mean an expression for the function $\boldsymbol{B}$, which may include roots of implicit equations, differentiations, and integrations. Though at the first sight, the benefit of such a "solution" may seem questionable, it occurs to be useful if one has a specific domain Ω and a function f at hand, see examples in the papers [4, 6], the whole paper [22] that treats the cases of functions f extremely difficult from an algebraic point of view, and other papers on the subject.

It appears that to solve Problem 1.1, one has to reformulate reasonings from [4] and [6] in geometric terms and observe that in such terms they work for a more general setting of the problem considered. For example, the integral (3.7) plays the role of the force function coming from $-\infty$ (see [4] for the definition in the case of BMO) in the general setting. However, the geometric essence of the matter is even more revealed in the example of the chordal domain. We remind the reader that a chordal domain is a type of foliation (see [4] for the definition) that consists of chords, i.e. segments that connect two points of $\partial\Omega_0$. In the case of the parabolic strip $g(t) = (t, t^2)$, the chordal domain could match $\boldsymbol{B}_f$ if and only if it satisfied the cup equation

$$\frac{\tilde{f}(b) - \tilde{f}(a)}{b - a} = \frac{\tilde{f}'(b) + \tilde{f}'(a)}{2}; \quad (a, a^2) \text{ and } (b, b^2) \text{ are the endpoints of a chord,}$$

and two special differential inequalities ("inequalities for the differentials") for each of its chord. In the general setting of Problem 1.1, the cup equation turns into

$$\begin{vmatrix} g_1'(a) & g_2'(a) & \tilde{f}'(a) \\ g_1'(b) & g_2'(b) & \tilde{f}'(b) \\ g_1(b) - g_1(a) & g_2(b) - g_2(a) & \tilde{f}(b) - \tilde{f}(a) \end{vmatrix} = 0; \quad g(a) \text{ and } g(b) \text{ are the endpoints of a chord,}$$

which has the following geometrical meaning: the tangent vectors to the curve $\gamma(t) = (g_1(t), g_2(t), \tilde{f}(t))$ at the points a and b lie in one two-dimensional plane with the vector $\gamma(a) - \gamma(b)$. The special differential inequalities (the so-called inequalities for the differentials) can also be re-stated in purely geometric terms (the triple product of $\gamma'(a)$, $\gamma(b) - \gamma(a)$, and the normal to γ at the point a should be negative; the same should be fulfilled with a and b interchanged) and then generalized to fit Problem 1.1.

In [4] the roots of $\tilde{f}'''$ played the main role. Indeed, the cups sit on the points where $\tilde{f}'''$ changes its sign from $+$ to $-$. In the general case, the function $\tilde{f}'''$ should be replaced by the torsion of the curve γ. One can see the traces of the torsion in formula (3.7). Moreover, now we see that Condition (3.5) is a straightforward generalization of the regularity condition from [4].

We recall that in [4] the problem was treated not in the full generality (we assumed that the roots of $\tilde{f}'''$ were well separated). This narrowed the list of local types of foliations. However, without such an assumption, the collection of figures is wider, see the forthcoming paper [6] for the general theory, and the example [22], where almost all figures from the general case appear. The latter paper also highlights the notation that becomes very important when there are lots of different figures (it appeared that a foliation corresponds to a special weighted graph). We only mention that all the figures are transferred to the general setting of Problem 1.1, as well as all the monotonicity lemmas for forces and tails (see [4] for definitions). However, in the general case there are some subtleties concerning different parametrizations of the curves g and γ. To formulate a right analog of a certian monotonicity lemma, one has to choose the right parametrization for it: sometimes it is more convenient to work in the natural parametrization of g, sometimes that of γ, sometimes it is useful to lay $g_1(t) = t$.

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