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Spectral Asymptotics for Problems with Integral Constraints

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Abstract—The eigenvalue problem for differential operators of arbitrary order with integral constraints is considered. The asymptotics of the eigenvalues is obtained. The results are applied to finding the asymptotics of the probability of small deviations for some detrended processes of nth order.

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1. INTRODUCTION

Consider the following eigenvalue problem:

$$(-1)^p u^{(2p)}(t) = \lambda^{(n,p)} u(t) + \mathcal{P}_{n-2p}(t), \qquad \int_0^1 t^i u(t)\,dt = 0, \quad i = 0,\dots,n-1, \tag{1.1}$$

where $n, p \in \mathbb{N}$, $n > 2p$, $\mathcal{P}_{n-2p}(t)$ is a polynomial with unknown coefficients of degree less than $n-2p$.

The aim of this paper is to find the asymptotics of the eigenvalues $\lambda_k^{(n,p)}$ as $k \to +\infty$, where $\lambda_k^{(n,p)}$ is the kth eigenvalue of problem (1.1). This problem arises in the study of the asymptotics of small deviations of Gaussian processes (see Sec. 3). For $p = 1$, this problem was considered in the paper [1] of Ai and Li. Note that second-order operators with integral conditions of more general form were considered in the monograph [2, Sec. 1.2] of Skubachevskii (see also the relevant bibliography given there).

Consider the following auxiliary boundary-value problem:

$$(-1)^p y^{(2n)}(t) = \lambda^{(n,p)} y^{(2n-2p)}(t), \qquad y^{(j)}(0) = y^{(j)}(1) = 0, \quad j = 0,\dots,n-1. \tag{1.2}$$

Problem (1.2) arises in the search for the sharp constant in the embedding theorem for the spaces $\mathring{W}_2^n(0,1) \hookrightarrow \mathring{W}_2^{n-p}(0,1)$:

$$\lambda_1^{(n,p)} = \min_{y \in \mathring{W}_2^n} \frac{\int_0^1 (y^{(n)}(x))^2\,dx}{\int_0^1 (y^{(n-p)}(x))^2\,dx}.$$

This constant was obtained by Janet [3] (see also [4]) for arbitrary $n \in \mathbb{Z}_+$ and $p = 1$. For an arbitrary $p \in \mathbb{N}$, the answer was stated in [5] without proof and in implicit terms (see also [6] for $p = 2$).

Lemma 1. *Problems* (1.1) *and* (1.2) *are equivalent, i.e., they have solutions for the same positive eigenvalues* $\lambda^{(n,p)}$; *further, if* $u(t)$ *is a solution of problem* (1.1) *and* $y(t)$ *is a solution of problem* (1.2), *then they are related by the equality* $u(t) = y^{(n)}(t)$.

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Proof. If we put $u(t) := y^{(n)}(t)$, then Eq. (1.2) takes the form

$$(-1)^p u^{(n)}(t) = \lambda^{(n,p)} u^{(n-2p)}(t),$$

which is equivalent to the equation in (1.1). Let us rewrite the boundary conditions (1.2) in terms of the function $u(t)$. By the Newton–Leibniz formula, we have

$$0 = y^{(n-1)}\Big|_0^1 = \int_0^1 y^{(n)}(t)\,dt = \int_0^1 v(t)\,dt,$$
$$0 = y^{(n-2)}\Big|_0^1 = \int_0^1 y^{(n-1)}(t)\,dt = ty^{(n-1)}\Big|_0^1 - \int_0^1 ty^{(n)}\,dt = -\int_0^1 tv(t)\,dt.$$

Similarly, all the other boundary conditions can be written in the form of the integral conditions from (1.1).

Conversely, we have the representation

$$y^{(k)}(t) = \frac{1}{(n-k-1)!}\int_0^t (t-s)^{(n-k-1)} u(s)\,ds;$$

therefore, from the integral conditions in (1.1), we obtain the boundary conditions in (1.2). □

The paper is organized as follows. In Sec. 2, we derive the equation for the eigenvalues $\lambda_k^{(n,p)}$ and construct their asymptotics. In Sec. 3, using the obtained results, we find the asymptotics of the probabilities of small deviations for some detrended processes.

In this paper, we use the following notation: $\mathfrak{V}[x_1,\dots,x_n]$ is the Vandermonde determinant; $J_k(x)$ is the Bessel function of the first kind of order k, and

$$[f(x)] := f(x) + O\left(\frac{1}{x}\right).$$

By even and odd functions we mean functions that are even or odd with respect to the point 1/2.

2. THE ASYMPTOTICS OF THE EIGENVALUES λ_k

Without loss of generality, we can assume that the solutions of problem (1.2) are either even or odd functions. Consider the even solution $y(t)$. Note that $y'(t)$ is an odd function satisfying the equation

$$(-1)^p (y'(t))^{(2n-1)} - \lambda^{(n,p)} (y'(t))^{(2n-2p-1)} = 0;$$

hence

$$(-1)^p (y'(t))^{(2n-2)} - \lambda^{(n,p)} (y'(t))^{(2(n-1-p))} = \text{const},$$

where the left-hand side is an odd and continuous function. Therefore, the constant on the right-hand side is 0. Thus, we see that the eigenvalue $\lambda^{(n,p)}$ corresponding to the even solution of the problem is $\lambda^{(n-1,p)}$, which corresponds to the odd solution of the problem.

Conversely, let us consider the odd solution $y(t)$ of problem (1.2) with $\lambda^{(n-1,p)}$. Obviously,

$$Y(t) := \int_0^t y(x)\,dx$$

is an even solution of the boundary-value problem

$$(-1)^p Y^{(2n)}(t) - \lambda^{(n-1,p)} Y^{(2n-2p)}(t) = 0, \qquad Y^{(j)}(0) = Y^{(j)}(1) = 0, \quad j = 0,\dots,n-1.$$

Thus, it suffices to consider only odd solutions of problem (1.2). Any such solution can be written as

$$y(t) = a_0 \sin(\xi_0(2t-1)) + a_1 \sin(\xi_1(2t-1)) + \dots + a_{p-1}\sin(\xi_{p-1}(2t-1)) + a_p(2t-1) + \dots + a_{n-1}(2t-1)^{2n-2p-1}, \tag{2.1}$$

where

$$\xi_j := \frac{\omega z^j}{2}, \quad j = 0, \dots, p-1, \qquad \omega = |\lambda^{(n,p)}|^{1/2p}, \qquad z = e^{i\pi/p}.$$

Substituting (2.1) into the boundary conditions from (1.2), we obtain a system of linear equations for a_j, $j = 0, \dots, n-1$.

For a nontrivial solution to exist, it is necessary that the determinant of the system be equal to zero. After cancellation by appropriate powers of 2, this determinant becomes

$$\begin{vmatrix} \sin(\xi_0) & \dots & \sin(\xi_{p-1}) & 1 & 1 & 1 & \dots & 1 \\ \xi_0\cos(\xi_0) & \dots & \xi_{p-1}\cos(\xi_{p-1}) & 1 & 3 & 5 & \dots & (2n-2p-1) \\ -\xi_0^2\sin(\xi_0) & \dots & -\xi_{p-1}^2\sin(\xi_{p-1}) & 0 & 3\cdot 2 & 5\cdot 4 & \dots & (2n-2p-1)(2n-2p-2) \\ -\xi_0^3\cos(\xi_0) & \dots & -\xi_{p-1}^3\cos(\xi_{p-1}) & 0 & 3! & 5\cdot 4\cdot 3 & \dots & \dots \\ \xi_0^4\sin(\xi_0) & \dots & \xi_{p-1}^4\sin(\xi_{p-1}) & 0 & 0 & 5\cdot 4\cdot 3\cdot 2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Denote this determinant by $\Delta_{n,p}$ and consider it as a function of the variables $(\xi_0, \dots, \xi_{p-1})$. Differentiating $\Delta_{n,p}$ with respect to each variable, we obtain

$$\frac{\partial^p \Delta_{n,p}}{\partial\xi_0 \dots \partial\xi_{p-1}} = \begin{vmatrix} \cos(\xi_0) & \dots & 1 & 1 & \dots & 1 \\ \cos(\xi_0) - \xi_0\sin(\xi_0) & \dots & 1 & 3 & \dots & (2n-2p-1) \\ -2\xi_0\sin(\xi_0) - \xi_0^2\cos(\xi_0) & \dots & 0 & 3\cdot 2 & \dots & (2n-2p-1)(2n-2p-2) \\ -3\xi_0^2\cos(\xi_0) + \xi_0^3\sin(\xi_0) & \dots & 0 & 3! & \dots & \dots \\ 4\xi_0^3\sin(\xi_0) + \xi_0^4\cos(\xi_0) & \dots & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Let us subject the resulting determinant to the following operations:

1) subtract the first row from the second, then subtract the doubled second row from the third, the tripled third row from the fourth, etc.;

2) expand the determinant $\Delta_{n,p}$ along the $(p+1)$th column, obtaining a determinant of order $(n-1)$;

3) take out common multipliers from each column, obtaining the recurrence relation

$$\frac{\partial^p}{\partial\xi_0 \dots \partial\xi_{p-1}} \Delta_{n,p} = 2\cdot 4\cdot 6 \cdots (2n-2p-2)\cdot \xi_0 \cdots \xi_{p-1} \cdot \Delta_{n-1,p}. \tag{2.2}$$

For $n = p$, we have

$$\Delta_{p,p} = \begin{vmatrix} \sin(\xi_0) & \dots & \sin(\xi_{p-1}) \\ \xi_0 \sin'(\xi_0) & \dots & \xi_{p-1}\sin'(\xi_{p-1}) \\ \vdots & \ddots & \vdots \\ \xi_0^{p-1}\sin^{(p-1)}(\xi_0) & \dots & \xi_{p-1}^{p-1}\sin^{(p-1)}(\xi_{p-1}) \end{vmatrix}.$$

Let us rewrite $\Delta_{p,p}$ in terms of the Bessel functions. Note that

$$\sin(x) = \sqrt{\frac{\pi}{2}}\, x^{1/2} J_{1/2}(x).$$

Subtracting the first row from the second, we can write

$$x\cos(x) - \sin(x) = -\sqrt{\frac{\pi}{2}} \cdot x^{3/2} J_{3/2}(x).$$

Similarly, subtracting from each row the appropriate linear combination of all previous rows and using formula [7, 8.463], we obtain the following expression for $\Delta_{p,p}$ (up to a multiplicative constant which is not important for our purposes):

$$\Delta_{p,p} = C_p \begin{vmatrix} \xi_0^{1/2} J_{1/2}(\xi_0) & \dots & \xi_{p-1}^{1/2} J_{1/2}(\xi_{p-1}) \\ \xi_0^{3/2} J_{3/2}(\xi_0) & \dots & \xi_{p-1}^{3/2} J_{3/2}(\xi_{p-1}) \\ \vdots & \ddots & \vdots \\ \xi_0^{(2p-1)/2} J_{(2p-1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2p-1)/2} J_{(2p-1)/2}(\xi_{p-1}) \end{vmatrix}.$$

Given $\Delta_{p,p}$, using relation (2.2), we will try to find $\Delta_{p+1,p}$. Let us multiply $\Delta_{p,p}$ by ξ_j, $j = 0, \dots, p-1$, and then integrate over each variable ξ_j from 0 to ξ_j. Using the recurrence relation between Bessel functions [7, 8.472.3], we can write

$$\Delta_{p+1,p} = C_{p+1} \begin{vmatrix} \xi_0^{3/2} J_{3/2}(\xi_0) & \dots & \xi_{p-1}^{3/2} J_{3/2}(\xi_{p-1}) \\ \xi_0^{5/2} J_{5/2}(\xi_0) & \dots & \xi_{p-1}^{5/2} J_{5/2}(\xi_{p-1}) \\ \vdots & \ddots & \vdots \\ \xi_0^{(2p+1)/2} J_{(2p+1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2p+1)/2} J_{(2p+1)/2}(\xi_{p-1}) \end{vmatrix}.$$

Thus, carrying out this operation $(n-p)$ times, we finally obtain

$$\Delta_{n,p} = C_n \begin{vmatrix} \xi_0^{(2n-2p+1)/2} J_{(2n-2p+1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-2p+1)/2} J_{(2n-2p+1)/2}(\xi_{p-1}) \\ \xi_0^{(2n-2p+3)/2} J_{(2n-2p+3)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-2p+3)/2} J_{(2n-2p+3)/2}(\xi_{p-1}) \\ \vdots & \ddots & \vdots \\ \xi_0^{(2n-1)/2} J_{(2n-1)/2}(\xi_0) & \dots & \xi_{p-1}^{(2n-1)/2} J_{(2n-1)/2}(\xi_{p-1}) \end{vmatrix}. \tag{2.3}$$

Theorem 1. *As $k \to \infty$, the following equality holds*:

$$\lambda_k^{(n,p)} = \left(\pi k + \frac{(2n-p-1)\pi}{2} + O(k^{-1})\right)^{2p}. \tag{2.4}$$

Proof. Consider $\Delta_{n,p}$ as a function of the single variable $\omega \in \mathbb{C}$ (recall that $\xi_j = \omega z^j/2$). Let us find the asymptotics of the roots $\Delta_{n,p}(\omega) = 0$ as $|\omega| \to \infty$. Note that $|\Delta_{n,p}(\omega)| = |\Delta_{n,p}(z\omega)|$; therefore, it suffices to consider $|\arg(\omega)| \le \pi/(2p)$.

In the angle $|\arg(\omega)| < \varphi_0 < \pi$, the Bessel function has the following uniform asymptotics at infinity (see [7, 8.451.1]):

$$J_{n+1/2}(\omega) = (-1)^n \sqrt{\frac{2}{\pi}} \frac{\sin^{(n)}(\omega)}{\omega^{1/2}} \left(1 + O\left(\frac{1}{\omega}\right)\right), \qquad |\omega| \to \infty.$$

Therefore, for $|\arg(\omega)| \le \pi/(2p)$, we have

$$\Delta_{n,p}(\omega) = C_n \left(\frac{2}{\pi}\right)^{\frac{p}{2}} \begin{vmatrix} \left(-\frac{\omega}{2}\right)^{n-p} \sin^{(n-p)}\left(\frac{\omega}{2}\right) & \dots & \left(-\frac{\omega}{2}\right)^{n-p} z^{(p-1)(n-p)} \sin^{(n-p)}\left(\frac{\omega}{2} z^{p-1}\right) \\ \vdots & \ddots & \vdots \\ \left(-\frac{\omega}{2}\right)^{n-1} \sin^{(n-1)}\left(\frac{\omega}{2}\right) & \dots & \left(-\frac{\omega}{2}\right)^{n-1} z^{(p-1)(n-1)} \sin^{(n-1)}\left(\frac{\omega}{2} z^{p-1}\right) \end{vmatrix} [1].$$

Note that, for $|\arg(\omega)| \le \pi/(2p)$, we have $\Im(\omega z^j) > a\omega > 0$ for $a > 0$ and $j = 0, \dots, p-1$; therefore,

$$\sin\left(\frac{\omega}{2} z^j\right) = -\frac{e^{-i(\omega/2)z^j}}{2i} \cdot [1], \qquad j = 1, \dots, p-1, \quad |\omega| \to \infty.$$

In all the columns, except the first, we replace the sines by the corresponding exponentials, obtaining

$$\Delta_{n,p} = \frac{C_n(-1)^{p-1}}{(2i)^p}\left(\frac{2}{\pi}\right)^{\frac{p}{2}}[1]$$
$$\times \begin{vmatrix} \left(-\frac{\omega}{2}\right)^{n-p}\left(i^{n-p}e^{\frac{i\omega}{2}} - (-i)^{n-p}e^{-\frac{i\omega}{2}}\right) & z^{n-p}\left(\frac{i\omega}{2}\right)^{n-p}e^{-i\frac{\omega}{2}z} & \dots & z^{(p-1)(n-p)}\left(\frac{i\omega}{2}\right)^{n-p}e^{-i\frac{\omega}{2}z^{p-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \left(-\frac{\omega}{2}\right)^{n-1}\left(i^{n-1}e^{\frac{i\omega}{2}} - (-i)^{n-1}e^{-\frac{i\omega}{2}}\right) & z^{n-1}\left(\frac{i\omega}{2}\right)^{n-1}e^{-i\frac{\omega}{2}z} & \dots & z^{(p-1)(n-1)}\left(\frac{i\omega}{2}\right)^{n-1}e^{-i)z^{p-1}} \end{vmatrix}.$$

Taking out common multipliers from the rows and columns, we obtain

$$\Delta_{n,p} = \frac{C_n(-1)^{p-1}}{(2\pi)^{(p/2)}i^p}\left(\frac{i\omega}{2}\right)^{p(2n-p-1)/2} e^{-i\omega/2}\cdots e^{-i(\omega/2)z^{p-1}} \cdot z^{(n-p)p(p-1)/2}$$
$$\times \begin{vmatrix} (-1)^{n-p}e^{i\omega} - 1 & 1 & \dots & 1 \\ (-1)^{n-p+1}e^{i\omega} - 1 & z & \dots & z^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n-1}e^{i\omega} - 1 & z^{p-1} & \dots & z^{(p-1)^2} \end{vmatrix} \cdot [1]. \tag{2.5}$$

Thus, the asymptotic solution of the equation $\Delta_{n,p}(\omega) = 0$ can be reduced to the solution of the equation

$$e^{i\omega}(-1)^{n-p} \cdot \mathfrak{V}[-1, z, \dots, z^{p-1}] = \mathfrak{V}[1, z, \dots, z^{p-1}] \cdot [1],$$

or

$$e^{i\omega}\left(1 + O\left(\frac{1}{\omega}\right)\right) = (-1)^{n-p}\frac{(z-1)\cdots(z^{p-1}-1)}{(z+1)\cdots(z^{p-1}+1)} = (-1)^{n-p}\frac{z-1}{z^{p-1}+1}\cdots\frac{z^{p-1}-1}{z+1}$$
$$= (-1)^{n-p}e^{i\pi/p}\cdots e^{i(p-1)\pi/p} = e^{i(2n-p-1)\pi/2}. \tag{2.6}$$

Standard arguments using the Rouché theorem (see, for example, [8, Chap. 2, Sec. 4]), show that all the roots of Eq. (2.6) of sufficiently large modulus are located in neighborhoods of the points $2\pi k + (2n-p-1)\pi/2$ with radius $O(k^{-1})$, exactly one root of the equation being in a neighborhood of each point. The roots of the equation $\Delta_{n-1,p}(\omega) = 0$ are near the points $2\pi k - \pi + (2n-p-1)\pi/2$, $k \to \infty$. Since the set $\{(\lambda_k^{(n,p)})^{1/(2p)}\}_{k\in\mathbb{N}}$ is the union of the positive roots of the determinants $\Delta_{n,p}$ and $\Delta_{n-1,p}$, we obtain the following asymptotics of the eigenvalues of problem (1.2):

$$\lambda_{k+k_0} = \left(\pi k + \frac{(2n-p-1)\pi}{2} + O(k^{-1})\right)^{2p}, \qquad k \to \infty,$$

where k_0 is an integer. It remains to show that $k_0 = 0$. To do this, let us use Jensen's theorem (see [9, Sec. 3.6]).

Denote $\delta := (2n-p-1)/2$. It follows from the equality $|\Delta_{n,p}(\omega)| = |\Delta_{n,p}(z\omega)|$, that the roots of the equation $\Delta_{n,p}(\omega) = 0$ have the form $\omega_k z^j$, $j = 0, \dots, 2p-1$, where the ω_k are the positive roots of this equation. In addition to these roots, there are extraneous roots $\omega = 0$ not corresponding to the eigenvalues. To exclude them, we will use the asymptotic behavior of the Bessel functions in a neighborhood of zero (see [7, 8.440]). As $\omega \to 0$, we obtain

$$\Delta_{n,p} = C_n \begin{vmatrix} \dfrac{(\omega/2)^{2n-2p+1}\cdot(1+o(1))}{2^{(2n-2p+1)/2}\Gamma(n-p+3/2)} & \dots & \dfrac{((\omega/2)z^{p-1})^{2n-2p+1}\cdot(1+o(1))}{2^{(2n-2p+1)/2}\Gamma(n-p+3/2)} \\ \vdots & \ddots & \vdots \\ \dfrac{(\omega/2)^{2n-1}\cdot(1+o(1))}{2^{(2n-1)/2}\Gamma(n+1/2)} & \dots & \dfrac{((\omega/2)z^{p-1})^{2n-1}\cdot(1+o(1))}{2^{(2n-1)/2}\Gamma(n+1/2)} \end{vmatrix}.$$

Hence

$$\frac{|\Delta_{n,p}(\omega)|}{\omega^{p(2n-p)}}\bigg|_{\omega=0} = \frac{C_n\cdot|\mathfrak{V}[1,z^2,\dots,z^{2(p-1)}]|\cdot 2^{-(3/2)p(2n-p)}}{\prod_{j=1}^{p}\Gamma(n-p+j+1/2)} = \frac{C_n\cdot p^{p/2}\cdot 2^{-(3/2)p(2n-p)}}{\prod_{j=1}^{p}\Gamma(n-p+j+1/2)}.$$

Consider the function

$$\widetilde{\Delta}_{n,p}(\omega) := \frac{\Delta_{n,p}(\omega)}{C_n\omega^{p(2n-p)}}\cdot\frac{\Delta_{n-1,p}(\omega)}{C_{n-1}\omega^{p(2n-2-p)}}. \tag{2.7}$$

Note that

$$\widetilde{\Delta}_{n,p}(0) = \frac{p^p\cdot 2^{-3p(2n-p-1)}}{\Gamma(n-p+1/2)\Gamma(n+1/2)\prod_{j=1}^{p-1}\Gamma^2(n-p+j+1/2)} \neq 0. \tag{2.8}$$

The zeros of the function $\widetilde{\Delta}_{n,p}(\omega)$ are asymptotically close to the zeros of the function $\Psi(\omega)$ (see [10, p. 8]),

$$\Psi(\omega) := \psi_\delta(\omega)\cdot\psi_\delta(\omega z)\cdot\dots\cdot\psi_\delta(\omega z^{p-1}),$$

where

$$\psi_\delta(\omega) = \prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{(\pi(n+\delta))^2}\right) = \frac{\Gamma^2(1+\delta)}{\Gamma(1+\delta+\omega/\pi)\Gamma(1+\delta-\omega/\pi)}.$$

Let us prove the existence of the uniform limit

$$\lim\frac{|\widetilde{\Delta}_{n,p}(\omega)|}{|\Psi(\omega)|} \qquad \text{for} \quad |\omega| = \pi\left(N+\delta+\frac{1}{2}\right), \quad N\to\infty.$$

Just as above, it suffices to restrict ourselves to the sector $|\arg(\omega)| \le \pi/(2p)$. It is known that (see [10, Lemma 1.3])

$$\psi_\delta(\omega) \sim \Gamma^2(1+\delta)\pi^{2\delta}\omega^{-2\delta-1}\cos\left(\omega-\pi\left(\delta+\frac{1}{2}\right)\right)$$

uniformly in $|\omega| = \pi(N+\delta+1/2)$, $N\to\infty$, in the given sector. Further, from formulas (2.5), (2.7), using (2.6), we obtain the following uniform asymptotic behavior as $|\omega|\to\infty$ and $|\arg(\omega)|\le\pi/(2p)$:

$$|\widetilde{\Delta}_{n,p}(\omega)| \sim (2\omega)^{-p(2n-p)}\left(\frac{2}{\pi}\right)^p\cdot|\mathfrak{V}[1,z,\dots,z^{p-1}]|^2\cdot|e^{-i\omega z}|\cdot\dots\cdot|e^{-i\omega z^{p-1}}|$$
$$\times|e^{\frac{i\omega}{2}}-e^{i(\delta\pi-\omega/2)}|\cdot|e^{\frac{i\omega}{2}}-e^{i(\delta\pi-\pi-\omega/2)}|.$$

For $j = 1\dots p-1$, we have

$$\left|\cos\left(\omega z^j-\pi\left(\delta+\frac{1}{2}\right)\right)\right| \sim \frac{1}{2}|e^{-i(\omega z^j-\pi(\delta+1/2))}| = \frac{1}{2}|e^{-i\omega z^j}| \qquad \text{as} \quad |\omega|\to\infty$$

uniformly in the given sector In addition,

$$|e^{i\omega/2}-e^{i(\delta\pi-\omega/2)}|\cdot|e^{i\omega/2}-e^{i(\delta\pi-\pi-\omega/2)}| = |e^{i\omega}-e^{i(2\delta\pi-\omega)}| = 2\left|\cos\left(\omega-\pi\left(\delta+\frac{1}{2}\right)\right)\right|.$$

Therefore, for $|\omega| = \pi(N+\delta+1/2)$ as $N\to\infty$,

$$\frac{|\widetilde{\Delta}_{n,p}(\omega)|}{|\Psi(\omega)|} \rightrightarrows \frac{2^{2p}\cdot|\mathfrak{V}[1,z,\dots,z^{p-1}]|^2}{\Gamma^{2p}(1+\delta)(2\pi)^{p(2n-p)}}. \tag{2.9}$$

By Jensen's theorem, we have

$$\prod_{k=1}^{N}\frac{(\pi(k+\delta))^{2p}}{\omega_k^{2p}} = \frac{|\Psi(0)|}{|\widetilde{\Delta}_{n,p}(0)|}\cdot\exp\biggl(\frac{1}{2\pi}\int_0^{2\pi}\ln\frac{|\widetilde{\Delta}_{n,p}(\pi(N+\delta+1/2)e^{i\varphi})|}{|\Psi(\pi(N+\delta+1/2)e^{i\varphi})|}\,d\varphi\biggr). \tag{2.10}$$

Using formulas (2.8), (2.9), and (2.10) we see that the infinite product

$$\prod_{k=1}^{\infty}\frac{(\pi(k+\delta))^{2p}}{\omega_k^{2p}}$$

converges. Hence we obtain $k_0=0$. □

3. APPLICATION TO THE ASYMPTOTICS OF SMALL DEVIATIONS

As was pointed out in the Introduction, problem (1.1) arises in the study of the asymptotics of small deviations $\mathbb{P}\{\|X_n\|_2<\varepsilon\}$ as $\varepsilon\to0$ for Gaussian processes $X_n(t)$, $t\in[0,1]$, of the form

$$X_n(t) := X(t) - \sum_{i=0}^{n-1} a_i t^i, \tag{3.1}$$

where the a_i are determined by the relations

$$\int_0^1 t^i X_n(t)\,dt = 0, \qquad i=0,\dots,n-1.$$

Here $X(t)$, $t\in[0,1]$, is a Gaussian process with zero mean ($\mathbb{E}X(t)\equiv0$) whose covariance function $G(s,t)=\mathbb{E}X(s)X(t)$ is the Green function of the following boundary-value problem:

$$Lu := (-1)^p u^{(2p)} = \lambda u + \text{ boundary conditions.} \tag{3.2}$$

The case $n=1$ (zero-mean processes) was intensively studied for different $X(t)$. In particular, results for the zero-mean Brownian motion $W_1(t)$ and the Brownian bridge $B_1(t)$ were obtained in [11], [12] (they correspond to the case $p=1$ and appropriate boundary conditions).

It is natural to regard the zero-mean process $X_1(t)$ as the projection $X(t)$ on the subspace of functions orthogonal to a constant in $L_2[0,1]$. If the projection on the linear functions is subtracted, then we obtain the so-called *detrended processes* $X_2(t)$. For the process $B_2(t)$, the asymptotics of small deviations was studied in the paper [13] of Ai and Li,. For an arbitrary n, the processes X_n are called *nth order detrended processes*. For the processes $W_n(t)$ and $B_n(t)$, the eigenvalues of the covariance operator were found in [1].

By the well-known Karhunen–Loève expansion (see, for example, [14, Sec. 12]), we have the following equality in distribution:

$$\|X_n\|_2^2 \overset{d}{=} \sum_{k=1}^{\infty}\mu_k\eta_k^2,$$

where the η_k, $k\in\mathbb{N}$, are the independent standard Gaussian random variables and the μ_k, $k\in\mathbb{N}$, are the eigenvalues of the integral operator with kernel

$$G_n(s,t) = \mathbb{E}X_n(s)X_n(t).$$

Note that

$$G_n(s,t) = \mathbb{E}\biggl(X(s)-\sum_{i=1}^{n-1}a_is^i\biggr)\biggl(X(t)-\sum_{i=1}^{n-1}a_it^i\biggr) = G(s,t)+\mathcal{P}_n(s,t),$$

where $\mathcal{P}_n(s,t)$ is a polynomial of degree at most $n-1$ in both variables. Then the equation for the eigenvalues is of the form

$$\mu u(t) = \int_0^1 u(s)(G(s,t) + \mathcal{P}_n(s,t))\,ds.$$

Applying the operator L to both sides of this equality and denoting $\lambda^{(n,p)} := \mu^{-1}$, we obtain Eq. (1.1). If we assume that $n \geq 2p$, we then see that $u(t)$ also satisfies the integral conditions from (1.1). In this case, the asymptotics of small deviations is independent of the original boundary conditions in (3.2).

To obtain the exact asymptotics of small deviations, we will use the comparison principle of W. Li.

Proposition 1 (see [15], [16]). *Let η_k be the sequence of independent standard Gaussian random variables, and let μ_k and $\widetilde{\mu}_k$ be two positive nonincreasing summable sequences such that $\prod \widetilde{\mu}_k/\mu_k < \infty$. Then*

$$\mathbb{P}\biggl\{\sum_{k=1}^{\infty} \mu_k \eta_k^2 < \varepsilon^2\biggr\} \sim \mathbb{P}\biggl\{\sum_{k=1}^{\infty} \widetilde{\mu}_k \eta_k^2 < \varepsilon^2\biggr\} \cdot \biggl(\prod_{k=1}^{\infty} \frac{\widetilde{\mu}_k}{\mu_k}\biggr)^{1/2}, \qquad \varepsilon \to 0.$$

For the approximant we take the sequence

$$\widetilde{\mu}_k := [\pi(k+\delta)]^{-2p}, \qquad k \in \mathbb{N}.$$

where $\delta = (2n-p-1)/2$. Then

$$\begin{aligned}\mathbb{P}\biggl\{\sum_{k=1}^{\infty} \mu_k \eta_k^2 < \varepsilon^2\biggr\} &\sim \mathbb{P}\biggl\{\sum_{k=1}^{\infty} \widetilde{\mu}_k \eta_k^2 < \varepsilon^2\biggr\} \cdot \biggl(\prod_{k=1}^{\infty} \frac{\widetilde{\mu}_k}{\mu_k}\biggr)^{1/2} \\ &= \mathbb{P}\biggl\{\sum_{k=1}^{\infty} \widetilde{\mu}_k \eta_k^2 < \varepsilon^2\biggr\} \cdot \prod_{k=1}^{\infty} \frac{\omega_k^p}{(\pi(k+\delta))^p}.\end{aligned}$$

The last product can be calculated from formulas (2.10), (2.8), (2.9). Using Theorem 6.2 from [17], we obtain the following theorem.

Theorem 2. *For the processes X_n, as $\varepsilon \to 0$,*

$$\mathbb{P}\{\|X_n\|_2 < \varepsilon\} \sim C\varepsilon^{\gamma} \exp\biggl(-\frac{2p-1}{2(2p\sin(\pi/(2p)))^{2p/(2p-1)}}\,\varepsilon^{-2/(2p-1)}\biggr),$$

where $\gamma = (1-2np+p^2)/(2p-1)$ and

$$C = \frac{(2p)^{1+\gamma/2+p/2} \cdot \pi^{(p-1)/2} \cdot \sin^{(1+\gamma)/2}(\pi/(2p))}{2^{p(2n-p-1/2)}\sqrt{2p-1} \cdot \mathfrak{V}[1, z, \dots, z^{p-1}]|} \cdot \frac{\Gamma^{-1/2}(n-p+1/2)\Gamma^{-1/2}(n+1/2)}{\prod_{j=1}^{p-1}\Gamma(n-p+j+1/2)}.$$

Remark. For $n = 2$, $p = 1$, this result was obtained in [13] without the knowledge of the exact value of the constant C. For the case $p = 1$ and an arbitrary n, the corresponding result in [1, Proposition 4.3] was also given for an unknown constant C. Besides, in that paper, the value of γ was calculated erroneously.

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