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THE EFFECT OF CURVATURE IN FRACTIONAL HARDY–SOBOLEV INEQUALITY INVOLVING THE SPECTRAL DIRICHLET LAPLACIAN

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ABSTRACT. We prove the attainability of the best constant in the fractional Hardy–Sobolev inequality with a boundary singularity for the spectral Dirichlet Laplacian. The main assumption is the average concavity of the boundary at the origin.

1. INTRODUCTION

In this paper we discuss the attainability of the best fractional Hardy–Sobolev constant $\mathcal{S}^{Sp}_{s,\sigma}(\Omega)$ in a $\mathcal{C}^1$–smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$:

$$\mathcal{S}^{Sp}_{s,\sigma}(\Omega) \cdot \||x|^{\sigma-s}u\|^2_{L_{2^*_\sigma}(\Omega)} \leqslant \langle(-\Delta)^s_{Sp}u,u\rangle, \quad u \in \widetilde{\mathcal{D}}^s(\Omega), \tag{1}$$

where $0 < \sigma < s < 1$ and $2^*_\sigma \equiv \frac{2n}{n-2\sigma}$. The operator in the right-hand side of (1) is the *spectral Dirichlet Laplacian*; the space $\widetilde{\mathcal{D}}^s(\Omega)$ is generated by its quadratic form (see Section 2).

In the case $0 \notin \overline{\Omega}$ the embedding $\widetilde{\mathcal{D}}^s(\Omega) \hookrightarrow L_{2^*_\sigma}(\Omega, |x|^{(\sigma-s)2^*_\sigma})$ is compact and $\mathcal{S}^{Sp}_{s,\sigma}(\Omega)$ is obviously attained. Through this paper we will consider the non-trivial case $0 \in \overline{\Omega}$.

In the local case $s = 1$ the inequality (1) coincides with

$$\mathcal{S}_\sigma(\Omega) \cdot \||x|^{\sigma-1}u\|^2_{L_{2^*_\sigma}(\Omega)} \leqslant \langle -\Delta u,u\rangle = \|\nabla u\|^2_{L_2(\Omega)}. \tag{2}$$

The attainability of the best constant $\mathcal{S}_\sigma(\Omega)$ is well-studied (even for the non-Hilbertian case), and the following facts are known:

- If $0 \in \Omega$, $\sigma \in [0,1]$, and $n \geqslant 3$, then $\mathcal{S}_\sigma(\Omega)$ does not depend on Ω. For $\sigma \in (0,1]$ the constant $\mathcal{S}_\sigma(\mathbb{R}^n)$ *is attained* on the family of functions

$$u_\varepsilon(x) := \left(\varepsilon + |x|^{\frac{2\sigma(n-2)}{n-2\sigma}}\right)^{1-\frac{n}{2\sigma}}$$

  ([12, 18]; in non-Hilbertian case see [1, 32] for $\sigma = 1$, [11] for $\sigma \in (0,1)$); thus $\mathcal{S}_\sigma(\Omega)$ *is not attained* if $\widetilde{\mathcal{D}}^1(\Omega) \neq \mathcal{D}^1(\mathbb{R}^n)$. If $\sigma = 0$, then $\mathcal{S}_\sigma$ *is not attained* even in $\mathbb{R}^n$ (see [13, Sec. 7.3]).

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- In the case $0 \in \partial\Omega$ the attainability of $\mathcal{S}_\sigma(\Omega)$ was proved for cones: if $\sigma \in (0,1)$, $n \geqslant 2$, and Ω is a cone in $\mathbb{R}^n$, then $\mathcal{S}_\sigma(\Omega)$ *is attained* ([7]; [27] in non-Hilbertian case).
- The case of a bounded domain Ω with $0 \in \partial\Omega$ is much more complex, and the answer depends on the behaviour of $\partial\Omega$ at the origin. In [9] it was shown that for $n \geqslant 4$ $\mathcal{S}_\sigma(\Omega)$ *is attained* if all principal curvatures of $\partial\Omega$ are negative at the origin. In [10] this condition was replaced by the negativity of the mean curvature of $\partial\Omega$ at the origin. In [6] these conditions were sufficiently weakened and the *attainability* was proved for all $n \geqslant 2$.

For $s \notin \mathbb{N}$ only a few results were established before. In [35] the *attainability* of $\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n)$ was shown for $s \in \left(0, \frac{n}{2}\right)$. For $s \in (0,1)$ the *attainability* of the best constant in $\mathbb{R}^n_+$ was shown for fractional Hardy–Sobolev inequalities with restricted Dirichlet and Neumann fractional Laplacians [21,25]. These inequalities differ from (1) by the choice of fractional Laplacian in the right-hand side.

In this paper we prove the following results for the inequality (1):

- In the case $0 \in \Omega$ and $\widetilde{\mathcal{D}}^s(\Omega) \neq \mathcal{D}^s(\mathbb{R}^n)$ the best constant $\mathcal{S}^{Sp}_{s,\sigma}(\Omega)$ *is not attained*. Moreover, if the domain Ω is star-shaped about the origin, then the corresponding Euler–Lagrange equation *does not have any non-trivial non-negative solutions*.
- The best constant $\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)$ *is attained*.
- In the case $0 \in \partial\Omega$ in a bounded Ω the best constant $\mathcal{S}^{Sp}_{s,\sigma}(\Omega)$ *is attained* under some geometrical assumptions on $\partial\Omega$ at the origin, analogous to the conditions from [6].

The short announcement of these results was given in [34].

The paper consists of nine sections. In Section 2 we give basic definitions and recall some properties of the spectral Dirichlet Laplacian (including the Stinga–Torrea extension). In Section 3 we prove the *unattainability* of $\mathcal{S}^{Sp}_{s,\sigma}(\Omega)$ in the case $0 \in \Omega$ together with the *non-existence of positive solutions* for the Euler–Lagrange equation in a star-shaped Ω. In Section 4 we derive estimates for the Green functions of some auxiliary problems. In Section 5 we prove the *attainability* of the best constant $\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)$. In Section 6 we formulate the assumptions on the behaviour of $\partial\Omega$ in a neighbourhood of the origin, which are sufficient for the *attainability* of $\mathcal{S}^{Sp}_{s,\sigma}(\Omega)$. The proof is based on the construction of a suitable trial function using the minimizer in $\mathbb{R}^n_+$. Estimates on this minimizer and on its Stinga–Torrea extension are given in Section 7: at first we derive the rough pointwise estimate of the minimizer and then we derive more accurate estimates, analogous to [6, Theorem 2.1]. Technical estimates used for the proof of the *attainability* in Ω are given in Sections 8, 9.

Notation. $x \equiv (x', x_n)$ is a point in $\mathbb{R}^n$ or in Ω; $y \equiv (y', y_n)$ is a point in the half-space

$$\mathbb{R}^n_+ := \{y \equiv (y', y_n) \in \mathbb{R}^n \mid y_n > 0\}.$$

We use the coordinates $X \equiv (x,t) \in \Omega \times \mathbb{R}_+$ dealing with the Stinga–Torrea extension from Ω and the coordinates $Y \equiv (y,z) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ dealing with the extension from $\mathbb{R}^n_+$.

We denote by $\mathbb{B}_r(x)$ and $\mathbb{S}_r(x)$ the sphere and the ball of radius r centered in x, respectively. For brevity we use the notation $\mathbb{B}_r := \mathbb{B}_r(0_n)$, $\mathbb{S}_r := \mathbb{S}_r(0_n)$, $\mathbb{B}^+_r := \mathbb{B}_r \cap \mathbb{R}^n_+$, and $\mathbb{K}^+_r := \mathbb{B}^+_{2r} \setminus \mathbb{B}^+_r$ (0_n stands for the origin in $\mathbb{R}^n$).

We fix a smooth cut-off function $\varphi_r(y)$ such that

$$\varphi_r(y) := \begin{cases} 1, & |y| < \frac{r}{2} \\ 0, & |y| > r \end{cases}, \quad |\nabla_y \varphi_r(y)| \leqslant \frac{C}{r}. \tag{3}$$

We use letter C to denote various positive constants depending on n, s, σ only. To indicate that C depends on some other parameters, we write $C(\dots)$. We also write $o_\varepsilon(1)$ to indicate a quantity that tends to zero as $\varepsilon \to 0$.

We use the notation $\widetilde{u}(y)$ and $\widetilde{w}(Y)$ for the odd reflections of $u(y)$ and its Stinga–Torrea extension $w(Y)$:

$$\widetilde{u}(y) := \begin{cases} u(y', y_n), & y_n \geqslant 0 \\ -u(y', -y_n), & y_n \leqslant 0 \end{cases}, \quad \widetilde{w}(Y) := \begin{cases} w(y', y_n, z), & y_n \geqslant 0, \\ -w(y', -y_n, z), & y_n \leqslant 0. \end{cases}$$

2. Preliminaries

Recall (see, for instance, [33, Secs. 2.3.3, 4.3.2]) that the Sobolev spaces $H^s(\mathbb{R}^n)$ and $\widetilde{H}^s(\Omega)$ are defined via the Fourier transform $\mathcal{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)dx$:

$$H^s(\mathbb{R}^n) = \left\{ u \in L_2(\mathbb{R}^n) \;\middle|\; \|u\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < +\infty \right\};$$

$$\widetilde{H}^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^n) \mid \operatorname{supp}(u) \subset \overline{\Omega} \right\}.$$

The fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ of a function $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is defined by the identity

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u(\xi)), \quad \langle (-\Delta)^s u, u \rangle = \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi. \tag{4}$$

The quadratic form in (4) is well-defined on $H^s(\mathbb{R}^n)$; thus the fractional Laplacian in $\mathbb{R}^n$ can be considered as a self-adjoint operator with the quadratic form (4) on $H^s(\mathbb{R}^n)$.

The spectral Dirichlet Laplacian $(-\Delta)^s_{Sp}$ is the s-th power of the conventional Dirichlet Laplacian in the sense of spectral theory. Its quadratic form in $\mathbb{R}^n$ coincides with (4), i.e. $(-\Delta)^s_{Sp} \equiv (-\Delta)^s$ in $\mathbb{R}^n$. In the case of $\Omega = \mathbb{R}^n_+$ the quadratic form is equal to

$$\langle (-\Delta)^s_{Sp} u, u \rangle := \int_{\mathbb{R}^n_+} |\xi|^{2s} |\widehat{\mathcal{F}}u(\xi)|^2 d\xi$$

with

$$\widehat{\mathcal{F}}u(\xi) := \frac{2}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-i\xi'\cdot x'} \sin(x_n \xi_n) dx;$$

for a bounded domain Ω,

$$\langle (-\Delta)^s_{Sp} u, u \rangle := \sum_{j=1}^{\infty} \lambda_j^s \langle u, \phi_j \rangle^2. \tag{5}$$

Here λ_j and ϕ_j are the eigenvalues and the eigenfunctions (orthonormalized in $L_2(\Omega)$) of the Dirichlet Laplacian on Ω, respectively.

Proposition 1 ([22, Theorem 2]). *Let $s \in (0,1)$. Then for $u(x) \in \widetilde{H}^s(\Omega)$ the following inequality holds:*

$$\langle (-\Delta)^s_{Sp} u, u \rangle \geqslant \langle (-\Delta)^s u, u \rangle. \tag{6}$$

If $u \not\equiv 0$, then (6) holds with a strict sign.

Inequality (1) for $s \in (0,1)$ (or even for $s \in \left(0, \frac{n}{2}\right)$) follows from (6) and the general theorem by V.P. Il'in [15, Theorem 1.2, (22)] about estimates of integral operators in weighted Lebesgue spaces.

Let $\Omega = \mathbb{R}^n$. If $\sigma = 0$, then inequality (1) reduces to the fractional Hardy inequality

$$\langle(-\Delta)^s u, u\rangle \geqslant \mathcal{S}_{s,0} \||x|^{-s} u\|^2_{L_2(\mathbb{R}^n)}, \tag{7}$$

and for $\sigma = s$ it reduces to the fractional Sobolev inequality

$$\langle(-\Delta)^s u, u\rangle \geqslant \mathcal{S}_{s,s} \|u\|^2_{L_{2^*_s}(\mathbb{R}^n)}. \tag{8}$$

The explicit values of $\mathcal{S}_{s,0}$ and $\mathcal{S}_{s,s}$ have been computed in [14] and [5], respectively. The explicit value of $\mathcal{S}_{s,\sigma}(\mathbb{R}^n)$ for arbitrary $\sigma \in (0,1)$ is unknown.

Thanks to (8), we can introduce the Hilbert spaces

$$\mathcal{D}^s(\mathbb{R}^n) := \left\{u \in L_{2^*_s}(\mathbb{R}^n) \mid \langle(-\Delta)^s u, u\rangle < \infty\right\},$$
$$\widetilde{\mathcal{D}}^s(\Omega) := \left\{u \in \mathcal{D}^s(\mathbb{R}^n) \mid u \equiv 0 \text{ outside of } \overline{\Omega}\right\},$$

both endowed with the scalar product $\langle(-\Delta)^s_{Sp} u, v\rangle$. The space $\mathcal{C}^\infty_0(\Omega)$ is dense in $\widetilde{\mathcal{D}}^s(\Omega)$. Obviously $\widetilde{\mathcal{D}}^s(\mathbb{R}^n_+) \cap L_2(\mathbb{R}^n_+) = \widetilde{H}^s(\mathbb{R}^n_+)$, and for any bounded Ω the Friedrichs inequality provides $\widetilde{\mathcal{D}}^s(\Omega) \equiv \widetilde{H}^s(\Omega)$.

We recall that the spectral Dirichlet Laplacian $(-\Delta)^s_{Sp}$ can be derived via the Stinga–Torrea extension [31]. It turns out that the Dirichlet problem

$$\mathcal{L}_s[w](X) \equiv -\mathrm{div}(t^{1-2s} \nabla_X w(x,t)) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+; \quad w|_{t=0} = u; \quad w|_{x \in \partial\Omega} = 0 \tag{9}$$

has a unique solution w_{sp} with finite energy

$$\mathcal{E}_s[w] := \int_0^{+\infty} \int_\Omega t^{1-2s} |\nabla_X w(x,t)|^2 dxdt. \tag{10}$$

In addition, the following relation holds in the sense of distributions:

$$(-\Delta)^s_{Sp} u(x) = C_s \frac{\partial w_{sp}}{\partial \nu_s}(x,0) := -C_s \lim_{t \to 0_+} t^{1-2s} \partial_t w_{sp}(x,t) \quad \text{with} \quad C_s := \tfrac{4^s \Gamma(1+s)}{2s \cdot \Gamma(1-s)}. \tag{11}$$

Moreover, w_{sp} is the minimizer of (10) over the space

$$\mathfrak{W}_s(\Omega) := \left\{w(X) \mid \mathcal{E}_s[w] < +\infty,\, w|_{t=0} = u,\, w|_{x \in \partial\Omega} = 0\right\},$$

and the quadratic form (5) can be expressed in terms of $\mathcal{E}_s[w_{sp}]$ (see, e.g., [23, (2.6)]):

$$\langle(-\Delta)^s_{Sp} u, u\rangle = C_s \mathcal{E}_s\left[w_{sp}\right]. \tag{12}$$

We refer to any function $w(X) \in \mathfrak{W}_s(\Omega)$ as an *admissible* extension of $u(x)$. Obviously, for any admissible extension w we have $\mathcal{E}_s[w] \geqslant \mathcal{E}_s[w_{sp}]$. As we noted above, for $\Omega = \mathbb{R}^n$ the spectral Dirichlet Laplacian coincides with the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$, and its extension (the Caffarelli–Silvestre extension) was introduced earlier in [2].

The attainability of $\mathcal{S}_{s,\sigma}(\Omega)$ is equivalent to the existence of a minimizer for the functional $\mathcal{I}_{\sigma,\Omega}$:

$$\mathcal{I}_{\sigma,\Omega}[u] := \frac{\langle(-\Delta)^s_{Sp} u, u\rangle}{\||x|^{\sigma-s} u\|^2_{L_{2^*_\sigma}(\Omega)}}. \tag{13}$$

A standard variational argument shows that each minimizer of (13) solves the following problem (up to multiplication by a constant):

$$(-\Delta)^s_{Sp} u(x) = \frac{|u|^{2^*_\sigma - 2} u(x)}{|x|^{(s-\sigma)2^*_\sigma}} \quad \text{in} \quad \Omega, \quad u \in \widetilde{\mathcal{D}}^s(\Omega). \tag{14}$$

The s-*Kelvin transform* in $\mathfrak{W}(\mathbb{R}^n)$ is defined by the formula

$$w^*(X) := \frac{1}{|X|^{n-2s}} w\Big(\frac{X}{|X|^2}\Big) \quad \forall X \equiv (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \setminus \{0_{n+1}\}. \tag{15}$$

The following properties hold for the s-Kelvin transform (see, e.g., [8, Proposition 2.6]):

$$\begin{cases} \mathcal{L}_s\left[w^*\right](X) = |X|^{-n-2s-2} \mathcal{L}_s\left[w\right]\left(\frac{X}{|X|^2}\right) & \forall X \equiv (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \setminus \{0_{n+1}\}, \\ \frac{\partial w^*}{\partial \nu_s}(x,0) \equiv |x|^{-n-2s} \frac{\partial w}{\partial \nu_s}\left(\frac{x}{|x|^2}, 0\right) & \forall x \in \mathbb{R}^n \setminus \{0_n\}. \end{cases}$$

The relation

$$\frac{\partial w^*}{\partial \nu_s}(x,0) \equiv |x|^{-n-2s} \frac{\partial w}{\partial \nu_s}\Big(\frac{x}{|x|^2}, 0\Big) = \frac{w^{2^*_\sigma - 1}\big(\frac{x}{|x|^2}, 0\big)|x|^{-n-2s}}{\big|\frac{x}{|x|^2}\big|^{(s-\sigma)2^*_\sigma}} = \frac{(w^*)^{2^*_\sigma - 1}(x,0)}{|x|^{2^*_\sigma(s-\sigma)}}$$

shows that the problem (14) is invariant under the s-Kelvin transform. This fact allows us to derive estimates of w near the origin and at infinity from each other.

In what follows, we need the following propositions:

Proposition 2 ([24, Theorem 3]). *Let $u(x) \in \widetilde{\mathcal{D}}^s(\Omega)$, $s \in (0,1)$. Then $|u(x)| \in \widetilde{\mathcal{D}}^s(\Omega)$ and*

$$\langle (-\Delta)^s_{Sp} u, u \rangle \geqslant \langle (-\Delta)^s_{Sp} |u|, |u| \rangle.$$

Moreover, if both the positive and the negative parts of u are non-trivial, then strict inequality holds.

The proof in [24] is given for bounded domains but works for unbounded domains without any changes.

Proposition 3 ([4, Lemma 2.6], [26, Proposition A.1]). *Let $s \in (0,1)$, let $u \not\equiv 0$, let $u(x) \in \widetilde{\mathcal{D}}^s(\Omega)$ or $u(x) \in \mathcal{D}^s(\mathbb{R}^n)$, and let $(-\Delta)^s_{Sp} u \geqslant 0$ hold in the sense of distributions. Then $u > 0$ for any compact $K \subset \Omega$ (or $K \subset \mathbb{R}^n$, respectively).*

According to Proposition 2 the substitution $u \to |u|$ decreases $\mathcal{I}_{\sigma,\Omega}$. Therefore, if u is a minimizer of (13), then the right-hand side of (14) is non-negative. Thus, the maximum principle from Proposition 3 shows that u preserves a sign.

Proposition 4 ([24, Proposition 3]). *Let $u(x) \in \widetilde{\mathcal{D}}^s(\Omega)$ and let $u_\rho(x) := \rho^{\frac{n-2s}{2}} u(\rho x)$. Then*

$$\langle (-\Delta)^s u, u \rangle = \lim_{\rho \to \infty} \langle (-\Delta)^s_{\Omega, Sp} u_\rho, u_\rho \rangle.$$

3. Non-existence results

In this section we consider the case $0 \in \Omega$.

Theorem 1. *Let $0 \in \Omega$ and $\widetilde{\mathcal{D}}^s(\Omega) \neq \mathcal{D}^s(\mathbb{R}^n)$.*

1. *The constant $\mathcal{S}^{Sp}_{s,\sigma}(\Omega)$ is not attained.*
2. *If Ω is star-shaped about 0, then the only non-negative solution of (14) is $u \equiv 0$.*

Proof. (1) In the local case $s=1$ this statement is well-known. We adapt it for the non-local case. At first we notice that $\mathcal{S}_{s,\sigma}(\mathbb{R}^n)$ can be approximated with $\mathcal{C}_0^\infty(\mathbb{R}^n)$ functions since such functions are dense in $\mathcal{D}^s(\mathbb{R}^n)$. Proposition 4 shows that for each $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have $u_\rho \in \widetilde{\mathcal{D}}^s(\Omega)$ for sufficiently large ρ and the following relation holds:

$$\lim_{\rho\to\infty} \mathcal{I}_{\sigma,\Omega}[u_\rho] = \mathcal{I}_{\sigma,\mathbb{R}^n}[u].$$

This means that $\mathcal{S}^{Sp}_{s,\sigma}(\Omega) \leqslant \mathcal{S}_{s,\sigma}(\mathbb{R}^n)$.

Let $\mathcal{S}^{Sp}_{s,\sigma}(\Omega)$ be attained on some $u \in \widetilde{\mathcal{D}}^s(\Omega)$. We extend u by zero to obtain a minimizer in $\mathbb{R}^n$. Indeed, the inequality (6) gives

$$\mathcal{I}_{\sigma,\mathbb{R}^n}[u] \leqslant \mathcal{S}^{Sp}_{s,\sigma}(\Omega) \leqslant \mathcal{S}_{s,\sigma}(\mathbb{R}^n),$$

which leads to a contradiction due to $\widetilde{\mathcal{D}}^s(\Omega) \neq \mathcal{D}^s(\mathbb{R}^n)$ and the maximum principle from Proposition 3.

(2) To prove the statement we invent a non-local variant of the Pohozhaev identity for $(-\Delta)^s_{Sp}$ (see [28] for $(-\Delta)^s$ in $\mathbb{R}^n$). Note that each solution of (14) has a singularity at the origin but is smooth outside the neighbourhood of the origin. Integrating by parts, we derive from (9) (here $\eta_\varepsilon(x) := 1-\varphi_\varepsilon(x)$, where $\varphi_\varepsilon(x)$ was introduced in (3)):

$$\begin{aligned}
0 &= C_s \int_0^{+\infty}\int_\Omega div\left(t^{1-2s}\nabla_X w(X)\right)\langle X, \nabla_X w(X)\rangle \eta_\varepsilon(x)dX \\
&= \int_\Omega \frac{u^{2^*_\sigma-1}(x)}{|x|^{(s-\sigma)2^*_\sigma}}\langle x, \nabla_x u(x)\rangle \eta_\varepsilon(x)dx \\
&\quad + C_s\int_0^{+\infty}\int_{\partial\Omega} t^{1-2s}\langle \nabla_x w(X), \vec{n}\rangle\langle x, \nabla_x w(X)\rangle \eta_\varepsilon(x)dX \\
&\quad - C_s\int_0^{+\infty}\int_\Omega t^{1-2s}|\nabla_X w(X)|^2\eta_\varepsilon(x)dX \\
&\quad - \frac{C_s}{2}\int_0^{+\infty}\int_\Omega t^{1-2s}\langle X, \nabla_X\left(|\nabla_X w(X)|^2\right)\rangle\eta_\varepsilon(x)dX \\
&\quad - C_s\int_0^{+\infty}\int_\Omega t^{1-2s}\langle\nabla_x w(X), \nabla_x\eta_\varepsilon(x)\rangle\langle X, \nabla_X w(X)\rangle dX \\
&=: B_1+B_2+B_3+B_4+B_5;
\end{aligned}$$

B_1 and B_2 contain ∇_x only since $w_t\big|_{x\in\partial\Omega}=0$ and $tw_t\big|_{t=0}=0$ due to (11). Further,

$$\begin{aligned}
B_1 &= \int_\Omega \sum_{i=1}^n \frac{\left[u^{2^*_\sigma}(x)\right]_{x_i}}{2^*_\sigma}\frac{x_i\eta_\varepsilon(x)}{|x|^{(s-\sigma)2^*_\sigma}}\,dx \\
&= \int_\Omega \frac{u^{2^*_\sigma}(x)}{2^*_\sigma}\sum_{i=1}^n\left(\frac{\eta_\varepsilon(x)}{|x|^{(s-\sigma)2^*_\sigma}} - \frac{2^*_\sigma(s-\sigma)\cdot x_i^2\eta_\varepsilon(x)}{|x|^{(s-\sigma)2^*_\sigma+2}}\right)dx + \int_\Omega \frac{u^{2^*_\sigma}(x)}{2^*_\sigma}\sum_{i=1}^n \frac{x_i\left[\eta_\varepsilon(x)\right]_{x_i}}{|x|^{(s-\sigma)2^*_\sigma}}\,dx \\
&= \left(\frac{n}{2^*_\sigma}-(s-\sigma)\right)\int_\Omega \frac{u^{2^*_\sigma}(x)(1-\varphi_\varepsilon(x))}{|x|^{(s-\sigma)2^*_\sigma}}\,dx - \int_\Omega \frac{u^{2^*_\sigma}(x)}{2^*_\sigma}\sum_{i=1}^n\frac{x_i\left[\varphi_\varepsilon(x)\right]_{x_i}}{|x|^{(s-\sigma)2^*_\sigma}}\,dx \\
&= -\frac{n-2s}{2}\langle(-\Delta)^s_{Sp}u, u\rangle - \frac{n-2s}{2}\int_\Omega\frac{u^{2^*_\sigma}(x)\varphi_\varepsilon(x)}{|x|^{(s-\sigma)2^*_\sigma}}\,dx \\
&\quad - \int_\Omega \frac{u^{2^*_\sigma}(x)}{2^*_\sigma}\sum_{i=1}^n\frac{x_i\left[\varphi_\varepsilon(x)\right]_{x_i}}{|x|^{(s-\sigma)2^*_\sigma}}\,dx.
\end{aligned}$$

Since $w|_{x\in\partial\Omega}=0$, vectors $\nabla_x w(X)$ and $\vec{n}$ are parallel, which gives

$$B_2 = C_s \int_0^{+\infty}\int_{\partial\Omega} t^{1-2s}\langle x,\, \vec{n}\rangle \cdot |\nabla_x w(X)|^2 \eta_\varepsilon(x) dX.$$

For B_3 we have

$$B_3 = -\langle(-\Delta)^s_{Sp} u, u\rangle + C_s \int_0^{+\infty}\int_{\Omega} t^{1-2s}|\nabla_X w(X)|^2 \varphi_\varepsilon(x) dX.$$

Integrating by parts in B_4 we obtain (using $t^{2-2s}|w_t(X)|^2\big|_{t=0}=0$)

$$\begin{aligned}
B_4 = &-\frac{C_s}{2}\int_0^{+\infty}\int_{\partial\Omega} t^{1-2s}\langle x,\, \vec{n}\rangle |\nabla_x w(X)|^2 \eta_\varepsilon(x) dX \\
&+ \frac{C_s(n-2s+2)}{2}\int_0^{+\infty}\int_{\Omega} t^{1-2s}|\nabla_X w(X)|^2\left(\eta_\varepsilon(x) + \langle x, \nabla_x \eta_\varepsilon(x)\rangle\right) dX \\
= &-\frac{C_s}{2}\int_0^{+\infty}\int_{\partial\Omega} t^{1-2s}\langle x,\, \vec{n}\rangle |\nabla_x w(X)|^2 \eta_\varepsilon(x)\, dX + \frac{n-2s+2}{2}\langle(-\Delta)^s_{Sp} u, u\rangle \\
&- \frac{C_s}{2}\int_0^{+\infty}\int_{\Omega} t^{1-2s}|\nabla_X w(X)|^2\left((n-2s+2)\varphi_\varepsilon(x) + \langle x, \nabla_x \varphi_\varepsilon(x)\rangle\right) dX.
\end{aligned}$$

Summing up, we get

$$\begin{aligned}
&\frac{C_s}{2}\int_0^{+\infty}\int_{\partial\Omega} t^{1-2s}\langle x,\, \vec{n}\rangle |\nabla_x w(X)|^2 dX \\
&\quad= \frac{C_s}{2}\int_0^{+\infty}\int_{\partial\Omega} t^{1-2s}\langle x,\, \vec{n}\rangle |\nabla_x w(X)|^2 \varphi_\varepsilon(x)\, dX \\
&\qquad+ \frac{n-2s}{2}\int_\Omega \frac{u^{2^*_\sigma}(x)\varphi_\varepsilon(x)}{|x|^{(s-\sigma)2^*_\sigma}}\, dx + \int_\Omega \frac{u^{2^*_\sigma}(x)}{2^*_\sigma}\sum_{i=1}^n \frac{x_i\,[\varphi_\varepsilon(x)]_{x_i}}{|x|^{(s-\sigma)2^*_\sigma}}\, dx \\
&\qquad+ \frac{C_s}{2}\int_0^{+\infty}\int_{\Omega} t^{1-2s}|\nabla_X w(X)|^2\left((n-2s)\varphi_\varepsilon(x) + \langle x, \nabla_x \varphi_\varepsilon(x)\rangle\right) dX \\
&\qquad- C_s\int_0^{+\infty}\int_{\Omega} t^{1-2s}\langle \nabla_x w(X), \nabla_x \varphi_\varepsilon(x)\rangle \langle X, \nabla_X w(X)\rangle\, dX.
\end{aligned}$$

The right-hand side of this equality tends to zero as $\varepsilon\to 0$; therefore, the left-hand side is zero. The assumption that Ω is star-shaped about 0 gives $\langle x,\, \vec{n}\rangle > 0$, thus $\nabla_x w = 0$ on $\partial\Omega$. Integrating by parts, we get

$$\begin{aligned}
0 &= \int_0^{+\infty}\int_\Omega \operatorname{div}(t^{1-2s}\nabla_X w(X)) dX \\
&= \int_\Omega \frac{u^{2^*_\sigma-1}(x)}{|x|^{(s-\sigma)2^*_\sigma}}\, dX + \lim_{t\to\infty}\int_\Omega \operatorname{div}(t^{1-2s}\nabla_X w(X))\, dx.
\end{aligned}$$

The second term in the right-hand side is zero (for more details see (46) and (47) in Section 6), but the first term is positive for $u(x)\geqslant 0, u\not\equiv 0$, a contradiction. □

Remark 1. The second statement of Theorem 1 is also valid in the case $0\in\partial\Omega$.

Below we assume that $0\in\partial\Omega$. We also can assume that $u(x)>0$ in Ω by Proposition 3 and $\||x|^{\sigma-s}u\|_{L_{2^*_\sigma}(\Omega)} = 1$ due to the invariance of (13) under dilations and multiplications by a constant.

4. Estimates on Green functions

The simplest problem involving the fractional Laplacian in $\mathbb{R}^n_+$ is

$$(-\Delta)^s_{Sp} u(y) = h(y) \quad \text{in} \quad \mathbb{R}^n_+. \tag{16}$$

The boundary value problem (BVP) (9) in $\mathbb{R}^n_+$ looks like

$$\mathcal{L}_s[w]\,(Y) \equiv -\operatorname{div}(z^{1-2s}\nabla_Y w\,(Y)) = 0 \quad \text{in} \quad \mathbb{R}^n_+ \times \mathbb{R}_+; \quad w|_{z=0} = u; \quad w|_{y_n=0} = 0. \tag{17}$$

The Stinga–Torrea extension $w\,(Y)$ can be derived from $h(y)$ by solving the BVP

$$\mathcal{L}_s[w]\,(Y) = 0 \quad \text{in} \quad \mathbb{R}^n_+ \times \mathbb{R}_+; \quad C_s \frac{\partial w}{\partial \nu_s}(y,0) = h(y); \quad w|_{y_n=0} = 0. \tag{18}$$

Lemma 1. *The Green functions of problems* (16)-(18) *are as follows:*

$$\textit{For } (18)\textit{:} \quad G_s(Y,\xi) := \frac{\widetilde{C}_{n,s}}{(|y-\xi|^2+z^2)^{\frac{n-2s}{2}}} \left(1 - \left[1 + \frac{4y_n\xi_n}{|y-\xi|^2+z^2}\right]^{\frac{2s-n}{2}}\right). \tag{19}$$

$$\textit{For } (17)\textit{:} \quad \Gamma_s(Y,\xi) := \frac{\widehat{C}_{n,s} z^{2s}}{(|y-\xi|^2+z^2)^{\frac{n+2s}{2}}} \left(1 - \left[1 + \frac{4y_n\xi_n}{|y-\xi|^2+z^2}\right]^{-\frac{n+2s}{2}}\right). \tag{20}$$

$$\textit{For } (16)\textit{:} \quad G_s(y,\xi) := G_s(y,0,\xi). \tag{21}$$

Proof. To obtain required Green functions, we consider the odd reflections $\widetilde{u}(y)$ and $\widetilde{w}\,(Y)$. Notice that $\widetilde{w}\,(Y)$ is the Stinga–Torrea extension of $\widetilde{u}(y)$ because of $w|_{y_n=0} = 0$. In [2] and [3, Remark 3.10] the Green functions in $\mathbb{R}^n$ were calculated for two problems: for the BVP

$$-\operatorname{div}(t^{1-2s}\nabla_X \widetilde{w}(X)) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \quad C_s \frac{\partial \widetilde{w}}{\partial \nu_s}(x,0) = \widetilde{h}(x)$$

we have the Green function $\widetilde{G}_s(X)$:

$$\widetilde{w}(X) = \int_{\mathbb{R}^n} \widetilde{G}_s(x-\xi,t)\widetilde{h}(\xi)d\xi \quad \text{with} \quad \widetilde{G}_s(X) := \frac{\widetilde{C}_{n,s}}{(x^2+t^2)^{\frac{n-2s}{2}}}; \tag{22}$$

for the BVP

$$-\operatorname{div}(t^{1-2s}\nabla_X \widetilde{w}(X)) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \quad \widetilde{w}|_{t=0} = \widetilde{u}$$

we have the Green function $\widetilde{\Gamma}_s(X)$:

$$\widetilde{w}(X) = \int_{\mathbb{R}^n} \widetilde{\Gamma}_s(x-\xi,t)\widetilde{u}(\xi)d\xi \quad \text{with} \quad \widetilde{\Gamma}_s(X) := \frac{\widehat{C}_{n,s} t^{2s}}{(x^2+t^2)^{\frac{n+2s}{2}}}. \tag{23}$$

The required representation (19) follows from (22) and from the identity

$$G_s(Y,\xi) = \widetilde{G}_s(y',y_n,t,\xi) - \widetilde{G}_s(y',-y_n,t,\xi) \quad \text{with} \quad y_n > 0.$$

Similarly, (20) follows from (23); the representation (21) is obvious. □

Lemma 2. *For any* $\mathfrak{b} \in [0,1]$ *the Green functions* $G_s(Y,\xi)$ *and* $\Gamma_s(Y,\xi)$ *satisfy the following estimates:*

$$G_s(Y,\xi) \leqslant \frac{C y_n^{\mathfrak{b}} \xi_n^{\mathfrak{b}}}{(|y-\xi|^2+z^2)^{\frac{n-2s+2\mathfrak{b}}{2}}} \quad \text{and} \quad \Gamma_s(Y,\xi) \leqslant \frac{C y_n^{\mathfrak{b}} \xi_n^{\mathfrak{b}} z^{2s}}{(|y-\xi|^2+z^2)^{\frac{n+2s+2\mathfrak{b}}{2}}}. \tag{24}$$

Also, $\nabla_Y G_s(Y,\xi)$ *can be estimated as follows:*

$$|\nabla_Y G_s(Y,\xi)| \leqslant \frac{C}{(|y-\xi|^2+z^2)^{\frac{n-2s+1}{2}}} \cdot \min\left(1, \frac{y_n\xi_n}{|y-\xi|^2+z^2} + \frac{\xi_n}{\sqrt{|y-\xi|^2+z^2}}\right). \tag{25}$$

Proof. The estimate for G_s follows from the interpolation of two inequalities:

$$G_s(Y,\xi) \leqslant \frac{C}{(|y-\xi|^2+z^2)^{\frac{n-2s}{2}}} \quad \text{and} \quad G_s(Y,\xi) \leqslant \frac{C y_n\xi_n}{(|y-\xi|^2+z^2)^{\frac{n-2s+2}{2}}}.$$

The first one is obvious, and the second one follows from the mean value theorem:

$$1 - \left[1 + \frac{4y_n\xi_n}{|y-\xi|^2+z^2}\right]^{\frac{2s-n}{2}} \leqslant \frac{C y_n\xi_n}{|y-\xi|^2+z^2}. \tag{26}$$

The estimate for Γ_s can be obtained in the same way.

The gradient $\nabla_Y G_s(Y,\xi)$ is given by the formulae (here $i \in \{1,\dots,n-1\}$)

$$\begin{pmatrix} \partial_z G_s(Y,\xi) \\ \partial_{y_i} G_s(Y,\xi) \\ \partial_{y_n} G_s(Y,\xi) \end{pmatrix} = C \cdot \begin{pmatrix} \frac{z}{(|y-\xi|^2+z^2)^{\frac{n-2s+2}{2}}} \left(1 - \left[1 + \frac{4y_n\xi_n}{|y-\xi|^2+z^2}\right]^{\frac{2s-n-2}{2}}\right) \\ \frac{y_i-\xi_i}{(|y-\xi|^2+z^2)^{\frac{n-2s+2}{2}}} \left(1 - \left[1 + \frac{4y_n\xi_n}{|y-\xi|^2+z^2}\right]^{\frac{2s-n-2}{2}}\right) \\ \frac{y_n-\xi_n}{(|y-\xi|^2+z^2)^{\frac{n-2s+2}{2}}} - \frac{y_n+\xi_n}{(|y'-\xi'|^2+|y_n+\xi_n|^2+z^2)^{\frac{n-2s+2}{2}}} \end{pmatrix};$$

therefore, the first part of (25) is obvious. The second part for $\partial_z G_s$ and $\partial_{y_i} G_s$ can be derived using the analogue of (26). Inequality for $\partial_{y_n} G_s$ follows from the inequality (recall that $\xi_n > 0$ and $y_n > 0$)

$$\begin{aligned} |\partial_{y_n} G_s(Y,\xi)| \leqslant{} & \frac{|y_n-\xi_n|}{(|y-\xi|^2+z^2)^{\frac{n-2s+2}{2}}} \left(1 - \left[1 + \frac{4y_n\xi_n}{|y-\xi|^2+z^2}\right]^{\frac{2s-n-2}{2}}\right) \\ & + \frac{2\xi_n}{(|y-\xi|^2+z^2)^{\frac{n-2s+2}{2}}} \end{aligned}$$

and the analogue of (26) for the expression in large brackets. □

5. Attainability of $\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)$

In this section we prove the existence of the minimizer for the functional (13) in the case $\Omega = \mathbb{R}^n_+$ and discuss its properties.

Theorem 2. *For* $\Omega = \mathbb{R}^n_+$ *there exists a minimizer of the functional* (13).

Proof. We follow the scheme in [27, Theorem 3.1] and based on the concentration-compactness principle of Lions [20]. Consider a minimizing sequence $\{u_k\}$ for (13). As was mentioned in Section 2, we can assume that $u_k(y) \geqslant 0$ and

$\||y|^{\sigma-s}u_k\|_{L_{2^*_\sigma}(\mathbb{R}^n_+)} = 1$. We also denote the Stinga–Torrea extensions as $w_k(Y)$ and define the functions $U_k(y)$ as

$$U_k(y) := \int_0^{+\infty} z^{1-2s}|\nabla_Y w_k(Y)|^2 dz. \tag{27}$$

Since $\{u_k\}$ is bounded in $\widetilde{\mathcal{D}}^s(\mathbb{R}^n_+)$, $w_k(Y)$ are uniformly bounded in $\mathfrak{W}_s(\mathbb{R}^n_+)$ as well as U_k and $||y|^{\sigma-s}u_k|^{2^*_\sigma}$ are uniformly bounded in $L_1(\mathbb{R}^n_+)$. Without loss of generality, we assume that:

- $u_k \rightharpoonup u$ in $\widetilde{\mathcal{D}}^s(\mathbb{R}^n_+)$;
- $\nabla_Y w_k \rightharpoonup \nabla_Y w$ in $L_2(\mathbb{R}^n_+ \times \mathbb{R}_+, z^{1-2s})$, and w is an admissible extension of u;
- $||y|^{\sigma-s}u_k|^{2^*_\sigma}$ weakly converges to a measure μ on $\overline{\mathbb{R}^n_+}$;
- U_k weakly converges to a measure $\mathcal{M}$ on $\overline{\mathbb{R}^n_+}$,

where $\overline{\mathbb{R}^n_+}$ is a one-point compactification of $\mathbb{R}^n_+$.

Embedding $\widetilde{\mathcal{D}}^s_{loc}(\mathbb{R}^n_+) \hookrightarrow L_{2^*_\sigma,loc}(\mathbb{R}^n_+\backslash\{0_n\})$ is compact due to $2^*_\sigma < 2^*_s$; thus, we have convergence $|y|^{\sigma-s}u_k \to |y|^{\sigma-s}u$ in $L_{2^*_\sigma,loc}(\mathbb{R}^n_+\backslash\{0_n\})$, which gives

$$\mu = ||y|^{\sigma-s}u|^{2^*_\sigma} + \alpha_0\boldsymbol{\delta_0}(y) + \alpha_\infty\boldsymbol{\delta_\infty}(y), \quad \alpha_0, \alpha_\infty \geqslant 0.$$

Here $\boldsymbol{\delta_0}(y)$ and $\boldsymbol{\delta_\infty}(y)$ are Dirac delta functions at the origin and at infinity, respectively.

Our next goal is to show that the measure $\mathcal{M}$ admits the estimate

$$\mathcal{M} \geqslant U + \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)\alpha_0^{\frac{2}{2^*_\sigma}}\boldsymbol{\delta_0}(y) + \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)\alpha_\infty^{\frac{2}{2^*_\sigma}}\boldsymbol{\delta_\infty}(y). \tag{28}$$

Obviously, it suffices to prove that $\mathcal{M}$ majorizes separately each term in the right-hand side of (28). The first estimate $\mathcal{M} \geqslant U$ follows from the weak convergence $\nabla_Y w_k\eta \rightharpoonup \nabla_Y w\eta$ in $L_2\left(\mathbb{R}^n_+ \times \mathbb{R}_+, z^{1-2s}\right)$ for any $\eta \in \mathcal{C}^\infty_0(\mathbb{R}^n_+)$ and from the weak lower semi-continuity of the weighted L_2-norm:

$$\begin{aligned}\int_{\mathbb{R}^n_+} \eta^2(y)d\mathcal{M} &\equiv \lim_{k\to\infty}\int_{\mathbb{R}^n_+} \eta^2(y)U_k(y)dy \\ &= \lim_{k\to\infty}\int_0^{+\infty}\int_{\mathbb{R}^n_+} z^{1-2s}|\nabla_Y w_k(Y)\cdot\eta(y)|^2 dY \\ &\geqslant \int_0^{+\infty}\int_{\mathbb{R}^n_+} z^{1-2s}|\nabla_Y w(Y)\cdot\eta(y)|^2 dY = \int_{\mathbb{R}^n_+}\eta^2(y)U(y)dy.\end{aligned} \tag{29}$$

To obtain the second estimate we use the trial function $\eta_\varepsilon(y) := \varphi_{2\varepsilon}(y)$:

$$\begin{aligned}&\int_{\mathbb{R}^n_+} \eta^2_\varepsilon(y)d\mathcal{M} \equiv \lim_{k\to\infty}\int_{\mathbb{R}^n_+} U_k\eta^2_\varepsilon(y)dy \\ &= \lim_{k\to\infty}\int_0^{+\infty}\int_{\mathbb{R}^n_+} z^{1-2s}|\nabla_Y\left[w_k(Y)\eta_\varepsilon(y)\right] - w_k(Y)\nabla_y\eta_\varepsilon(y)|^2 dY \\ &= \lim_{k\to\infty}\int_0^{+\infty}\int_{\mathbb{R}^n_+}\Big[z^{1-2s}|\nabla_Y\left[w_k(Y)\eta_\varepsilon(y)\right]|^2 \\ &- 2z^{1-2s}\nabla_y w_k(Y)\nabla_y\eta_\varepsilon(y)w_k(Y)\eta_\varepsilon(y) \\ &+ z^{1-2s}|w_k(Y)\nabla_y\eta_\varepsilon(y)|^2\Big]dY =: D_1 - D_2 + D_3.\end{aligned} \tag{30}$$

To estimate D_1 we use the Hardy–Sobolev inequality (1):

$$D_1 \geqslant \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) \cdot \lim_{k\to\infty} \||y|^{\sigma-s} u_k \varphi_{2\varepsilon}\|^2_{L_{2^*_\sigma}(\mathbb{R}^n_+)} \geqslant \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) \alpha_0^{\frac{2}{2^*_\sigma}}. \tag{31}$$

To estimate D_3 we have to pass to the limit under the integral side:

Lemma 3. *The following equality holds:*

$$D_3 = \int_0^{+\infty} \int_{\mathbb{R}^n_+} z^{1-2s} w^2(Y) |\nabla_y \eta_\varepsilon(y)|^2 dY. \tag{32}$$

Proof. Let $\delta \in (0,1)$. We split the integral into three parts:

$$D_3 = a_k + b_k + c_k := \left(\int_0^\delta + \int_\delta^{\frac{1}{\delta}} + \int_{\frac{1}{\delta}}^{+\infty} \right) \int_{\mathbb{R}^n_+} z^{1-2s} w_k^2(Y) |\nabla_y \eta_\varepsilon(y)|^2 dY.$$

Since the sequence w_k is bounded in $W^1_2\left(\mathbb{K}^+_\varepsilon \times \left[\delta, \frac{1}{\delta}\right]\right)$, $w_k \to w$ in $L_2\left(\mathbb{K}^+_\varepsilon \times \left[\delta, \frac{1}{\delta}\right]\right)$, and thus we can pass to the limit in b_k. To complete the proof, it suffices to show that

$$a_k, c_k < C(\varepsilon) \cdot \delta^{1-s}. \tag{33}$$

To prove (33) for a_k we use the Green function (20):

$$\begin{aligned} w_k(Y) &= \int_{\mathbb{R}^n_+} u_k(\xi) \Gamma_s(Y,\xi) d\xi = \left(\int_{|y-\xi|>1} + \int_{|y-\xi|\leqslant 1} \right) u_k(\xi) \Gamma_s(Y,\xi) d\xi \\ &=: w_{1k}(Y) + w_{2k}(Y), \\ a_k &\leqslant 2 \int_0^\delta \int_{\mathbb{R}^n_+} z^{1-2s} \left[w_{1k}^2(Y) + w_{2k}^2(Y) \right] \cdot |\nabla_y \eta_\varepsilon(y)|^2 \, dY =: a_{1k} + a_{2k}. \end{aligned}$$

Using (24) for $\mathfrak{b} = 0$, $|\nabla_y \varphi_{2\varepsilon}| \leqslant \frac{c}{\varepsilon}$, and the Cauchy–Bunyakovsky–Schwarz inequality we get

$$\begin{aligned} a_{1k} &\leqslant \frac{C}{\varepsilon^2} \int_0^\delta z^{1-2s} \int_{|y|<2\varepsilon} \left(\int_{|y-\xi|>1} \frac{u_k(\xi) z^{2s}}{(|y-\xi|^2+z^2)^{\frac{n+2s}{2}}} d\xi \right)^2 dY \\ &\leqslant C \frac{\delta^{2+2s}}{\varepsilon^2} \int_{|y|<2\varepsilon} \left(\int_{|y-\xi|>1} \frac{u_k(\xi)|\xi|^{-s}|\xi|^s}{|y-\xi|^{n+2s}} d\xi \right)^2 dy \\ &\leqslant C(\varepsilon) \delta^{2+2s} \||y|^{-s} u_k\|^2_{L_2(\mathbb{R}^n_+)} \int_1^{+\infty} r^{-n-2s-1} dr. \end{aligned}$$

Similarly, we get the estimate

$$\begin{aligned} a_{2k} \leqslant \frac{C}{\varepsilon^2} \int_0^\delta z^{-s} \int_{|y|<2\varepsilon} \left(\int_{|y-\xi|\leqslant 1} \frac{u_k(\xi) z^{2s+\frac{1-s}{2}} d\xi}{(|y-\xi|^2+z^2)^{\frac{n+2s}{2}}} \right)^2 dY \\ \leqslant C \frac{\delta^{1-s}}{\varepsilon^2} \int_{|y|<2\varepsilon} \left(\int_{|y-\xi|\leqslant 1} \frac{u_k(\xi) d\xi}{|y-\xi|^{n-\frac{1-s}{2}}} \right)^2 dy. \end{aligned}$$

We estimate the integrand from the right-hand side as follows:

$$\left(\int_{|y-\xi|\leqslant 1}\frac{u_k(\xi)}{|y-\xi|^{n-\frac{1-s}{2}}}d\xi\right)^2\leqslant C\left(\int_{|y-\xi|\leqslant 1}\frac{u_k(\xi)-u_k(y)}{|y-\xi|^{n-\frac{1-s}{2}}}d\xi+u_k(y)\right)^2$$
$$\leqslant C\left(\int_{|y-\xi|\leqslant 1}\frac{u_k(\xi)-u_k(y)}{|y-\xi|^{n-\frac{1-s}{2}}}d\xi\right)^2+Cu_k^2(y)$$
$$\leqslant C\int_{|y-\xi|\leqslant 1}\frac{|u_k(\xi)-u_k(y)|^2}{|y-\xi|^{n+2s}}d\xi\cdot\int_0^1 r^s dr+Cu_k^2(y).$$

Finally, using (6) we get

$$a_{2k}\leqslant C(\varepsilon)\delta^{1-s}\left(\langle(-\Delta)^s_{Sp}u_k,u_k\rangle\cdot\int_0^1 r^s dr+\||y|^{-s}u_k\|^2_{L_2(\mathbb{R}^n_+)}\right).$$

To prove (33) for c_k we use (24) with $\mathfrak{b}=1$:

$$c_k\leqslant\frac{C}{\varepsilon^2}\int_{\frac{1}{\delta}}^{+\infty}z^{1-2s}\int_{|y|<2\varepsilon}\left(\int_{\mathbb{R}^n_+}\frac{u_k(\xi)z^{2s}|y_n|\xi_n}{(|y-\xi|^2+z^2)^{\frac{n+2s+2}{2}}}\,d\xi\right)^2 dY$$
$$\leqslant C(\varepsilon)\||y|^{-s}u_k\|^2_{L_2(\mathbb{R}^n_+)}\int_{\frac{1}{\delta}}^{+\infty}z^{1+2s}\left(\int_0^{+\infty}\frac{r^{n+1+2s}}{(r^2+z^2)^{n+2s+2}}\,dr\right)dz$$
$$\leqslant C(\varepsilon)\||y|^{-s}u_k\|^2_{L_2(\mathbb{R}^n_+)}\int_{\frac{1}{\delta}}^{+\infty}z^{-1-n}dz=C(\varepsilon)\delta^n\||y|^{-s}u_k\|^2_{L_2(\mathbb{R}^n_+)}.$$

Thus, the estimate (33) is completely proved and we get (32). □

Lemma 3 implies that

$$D_3\leqslant\frac{C}{\varepsilon^2}\int_0^{+\infty}\int_{|y|<2\varepsilon}z^{1-2s}|w(Y)|^2dY. \tag{34}$$

For $y_n\leqslant 2\varepsilon$, using the inequality

$$|w(Y)|^2=\left(\int_0^{y_n}\frac{\partial w(y',t,z)}{\partial y_n}\,dt\right)^2\leqslant 2\varepsilon\int_0^{2\varepsilon}\left(\frac{\partial w}{\partial y_n}\right)^2dt,$$

we obtain

$$D_3\leqslant\int_0^{+\infty}\int_{|y|<2\sqrt{2}\varepsilon}z^{1-2s}\left(\frac{\partial w}{\partial y_n}\right)^2dY=o_\varepsilon(1)\cdot\mathcal{E}_s[w].$$

To estimate D_2 we use the Cauchy–Bunyakovsky–Schwarz inequality

$$|D_2|\leqslant C\sqrt{D_1\cdot D_3}=o_\varepsilon(1). \tag{35}$$

To sum up, we have transformed (30) into

$$\int_{\mathbb{R}^n_+}\varphi^2_{2\varepsilon}\,d\mathcal{M}\equiv\lim_{k\to+\infty}\int_{\mathbb{R}^n_+}U_k\varphi^2_{2\varepsilon}\,dy\geqslant\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)\alpha_0^{\frac{2}{2^*_\sigma}}+o_\varepsilon(1),$$

which gives $\mathcal{M}\geqslant\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)\alpha_0^{\frac{2}{2^*_\sigma}}\boldsymbol{\delta}_{\mathbf{0}}(y)$.

To derive the estimate at infinity, we put $\eta_\varepsilon(y) := 1 - \varphi_{\frac{2}{\varepsilon}}(y)$ and write (30) for it. The estimate for D_1 is similar to (31). To estimate D_3 we use the analogue of Lemma 3:

$$D_3 \leqslant \varepsilon^2 \int_0^{+\infty} \int_{\mathbb{K}_{\frac{1}{\varepsilon}}} z^{1-2s} |w(Y)|^2 \, dY.$$

In spherical coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ we have $w = 0$ for $\theta_{n-1} = 0$; thus

$$|w(Y)|^2 = |w(r, \theta_1, \ldots, \theta_{n-1})|^2 = \left(\int_0^{\theta_{n-1}} \frac{\partial w(y', t, z)}{\partial \theta_{n-1}} \, dt \right)^2 \leqslant \pi \int_0^{\pi} \left(\frac{\partial w}{\partial \theta_{n-1}} \right)^2 dt,$$

$$D_3 \leqslant \pi^2 \varepsilon^2 \int_0^{+\infty} \int_{\mathbb{K}_{\frac{1}{\varepsilon}}} z^{1-2s} \left(\frac{\partial w}{\partial \theta_{n-1}} \right)^2 dY \leqslant \frac{4\pi^2 \varepsilon^2}{\varepsilon^2} \int_0^{+\infty} \int_{\mathbb{K}_{\frac{1}{\varepsilon}}} z^{1-2s} |\nabla_y w|^2 dY$$
$$= o_\varepsilon(1) \cdot \mathcal{E}_s[w].$$

Further arguments are similar to the estimate at the origin. The inequality (28) is proved.

The end of the proof is rather standard. By dilations and multiplications on a suitable constant one can achieve

$$\| |y|^{\sigma - s} u_k \|_{L_{2^*_\sigma}(\mathbb{B}_1^+)} = \| |y|^{\sigma - s} u_k \|_{L_{2^*_\sigma}(\mathbb{R}^n_+ \setminus \mathbb{B}_1^+)} = \frac{1}{2}. \tag{36}$$

From (28) and the fact that w is an admissible extension of u we get

$$\begin{aligned}
\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) &\left(\| |y|^{\sigma - s} u \|^2_{L_{2^*_\sigma}(\mathbb{R}^n_+)} + \alpha_0^{\frac{2}{2^*_\sigma}} + \alpha_\infty^{\frac{2}{2^*_\sigma}} \right) \\
&\leqslant \langle (-\Delta)^s_{Sp} u, u \rangle + \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) \alpha_0^{\frac{2}{2^*_\sigma}} + \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) \alpha_\infty^{\frac{2}{2^*_\sigma}} \\
&\leqslant \int_{\mathbb{R}^n_+} 1 d\mathcal{M} = \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) \left(\int_{\mathbb{R}^n_+} 1 d\mu \right)^{\frac{2}{2^*_\sigma}} \\
&= \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) \left(\| |y|^{\sigma - s} u \|^{2^*_\sigma}_{L_{2^*_\sigma}(\mathbb{R}^n_+)} + \alpha_0 + \alpha_\infty \right)^{\frac{2}{2^*_\sigma}},
\end{aligned} \tag{37}$$

which can be true only if two of three terms from the right-hand side vanish. The relation (36) keeps only the possibility that $\alpha_0 = \alpha_\infty = 0$; i.e. u is a minimizer of (13). □

Remark 2. The minimizer existence for any cone in $\mathbb{R}^n$ can be proved in a similar way.

We denote the obtained minimizer in $\mathbb{R}^n_+$ by $\Phi(y)$ and its Stinga–Torrea extension by $\mathcal{W}(Y)$. Without loss of generality, we can assume that $\| |y|^{\sigma - s} \Phi \|_{L_{2^*_\sigma}(\mathbb{R}^n_+)} = 1$; therefore we have $\mathcal{E}_s [\mathcal{W}] = \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)$.

Lemma 4. *$\Phi(y)$ and $\mathcal{W}(Y)$ are radial in y' and positive for $y_n > 0$.*

Proof. The positivity of $\Phi(y)$ and $\mathcal{W}(Y)$ was proved at the end of Section 2. To prove the first part we show that a non-trivial partial Schwarz symmetrization on

y' (we denote the symmetrization of u as $u^\star$) decreases (13):

$$\mathcal{I}_{\sigma,\Omega}[u] = \frac{\mathcal{E}_s\left[w_{sp}\right]}{\||x|^{\sigma-s}u\|^2_{L_{2^*_\sigma}(\Omega)}} \overset{*}{\geqslant} \frac{\mathcal{E}_s\left[w^\star_{sp}\right]}{\||x|^{\sigma-s}u\|^2_{L_{2^*_\sigma}(\Omega)}} \overset{**}{>} \frac{\mathcal{E}_s\left[w^\star_{sp}\right]}{\||x|^{\sigma-s}u^\star\|^2_{L_{2^*_\sigma}(\Omega)}} \overset{***}{\geqslant} \mathcal{I}_{\sigma,\Omega}[u^\star].$$

The inequality (*) is provided by the fact that $\mathcal{E}_s\left[w_{sp}\right]$ does not increase under symmetrization (see [16, Theorem 2.31, p. 83] for the Steiner symmetrization; partial Schwarz symmetrization can be achieved as the limit of Steiner symmetrizations). The inequality (**) follows from [19, Theorem 3.4]. The fact that $w^\star_{sp}$ is an admissible extension for $u^\star$ gives (***). □

Remark 3. Minimizer of (13) with $\||y|^{\sigma-s}\Phi\|_{L_{2^*_\sigma}(\mathbb{R}^n_+)} = 1$ is not unique. Indeed, the functional (13) is invariant with respect to dilations and multiplications by constant. Compositions of these transformations that keep $\||y|^{\sigma-s}\Phi\|_{L_{2^*_\sigma}(\mathbb{R}^n_+)}$ norm give us multiple minimizers.

For the further discussion, we fix some minimizer and study its behaviour at the origin and at infinity:

Lemma 5. *The minimizer $\Phi(y)$ and its Stinga–Torrea extension $\mathcal{W}(Y)$ admit the following estimates:*

$$\Phi(y) \leqslant \frac{Cy_n}{1+|y|^{n-2s+2}},\ y \in \mathbb{R}^n_+; \quad \mathcal{W}(Y) \leqslant \frac{Cy_n}{1+|Y|^{n-2s+2}},\ Y \in \mathbb{R}^n_+ \times \mathbb{R}_+; \tag{38}$$

$$\mathcal{V}(y) := \int_0^{+\infty} z^{1-2s}|\nabla_Y \mathcal{W}(Y)|^2 dz \leqslant \frac{C}{1+|y|^{2n-2s+2}},\ y \in \mathbb{R}^n_+, \tag{39}$$

where constants C depend on n, s, σ, and on the choice of the minimizer Φ.

The proof of Lemma 5 is given in Section 7.

6. Attainability of $\mathcal{S}^{Sp}_{s,\sigma}(\Omega)$

We assume that in a small ball $\mathbb{B}_{r_0}$ (centered at the origin) the surface $\partial\Omega$ is parametrized by the equation $x_n = F(x')$, where $F \in \mathcal{C}^1$, $F(0_{n-1}) = 0$, and $\nabla_{x'}F(0_{n-1}) = 0_{n-1}$. Note that no assumptions on $\partial\Omega$ outside $\mathbb{B}_{r_0}$ are imposed.

Following [6], we assume that $\partial\Omega$ is *average concave at the origin*: for small $\tau > 0$,

$$f(\tau) := \frac{1}{|\mathbb{S}^{n-2}_\tau|}\int_{\mathbb{S}^{n-2}_\tau} F(y')\,dy' < 0. \tag{40}$$

Obviously, $f \in \mathcal{C}^1$ for small τ. We also assume that f is *regularly varying* at the origin with the exponent $\alpha \in [1, n-2s+3)$: for any $d > 0$,

$$\lim_{\tau\to 0}\frac{f(d\tau)}{f(\tau)} = d^\alpha. \tag{41}$$

It is well-known (see, e.g., [29, Secs. 1.1, 1.2]) that (41) entails $f(\tau) := -\tau^\alpha\psi(\tau)$ with the slowly varying function $\psi(\tau)$ (SVF). Note that for $\alpha = 1$ the condition $F \in \mathcal{C}^1$ implies that $\lim_{\tau\to 0}\psi(\tau) = 0$.

We also introduce the functions

$$f_1(\tau) := \frac{1}{|\mathbb{S}_\tau^{n-2}|}\int_{\mathbb{S}_\tau^{n-2}} F^2(y')\,dy'; \quad f_2(\tau) := \frac{1}{|\mathbb{S}_\tau^{n-2}|}\int_{\mathbb{S}_\tau^{n-2}} |\nabla_{y'}F(y')|^2\,dy';$$

$$f_3(\tau) := \frac{1}{|\mathbb{S}_\tau^{n-2}|}\int_{\mathbb{S}_\tau^{n-2}} |\nabla_{y'}F(y')|\,dy'$$

and assume that the following condition is fulfilled:

$$\lim_{\tau\to 0}\frac{f_2(\tau)}{f(\tau)}\tau = 0. \tag{42}$$

Remark 4. If $\partial\Omega \in \mathcal{C}^2$, then the negativity of its mean curvature at the origin implies assumptions (40)-(42) with $\alpha = 2$ (see [6, Remark 1]). We also emphasize that these assumptions admit the absence of mean curvature ($\alpha < 2$) or its vanishing ($\alpha > 2$).

Remark 5. It was shown in [6, Sec. 4, (17)] that (42) implies that

$$f_1(\tau) \leqslant C\tau|f(\tau)|\cdot o_\tau(1). \tag{43}$$

Theorem 3. *Let $\partial\Omega$ satisfy* (40)-(42). *Then the minimizer of* (13) *exists; i.e. the problem* (14) *has a positive solution in* Ω.

Proof. The scheme of the proof is the same as in Theorem 2. Consider a minimizing sequence $\{u_k\}$ for (13). We denote the Stinga–Torrea extensions as $w_k(Y)$ and define functions $U_k(y)$ via (27). As before, $U_k \in L_1(\Omega)$ and $||x|^{\sigma-s}u_k|^{2^*_\sigma} \in L_1(\Omega)$, and we can also assume that:

- $u_k \geqslant 0$, $u_k \rightharpoonup u$ in $\widetilde{\mathcal{D}}^s(\Omega)$;
- $\nabla_X w_k \rightharpoonup \nabla_X w$ in $L_2(\Omega\times\mathbb{R}_+, t^{1-2s})$ and w is an admissible extension of u;
- $||x|^{\sigma-s}u_k|^{2^*_\sigma}$ weakly converges to a measure μ on $\overline{\Omega}$;
- U_k weakly converges to a measure $\mathcal{M}$ on $\overline{\Omega}$.

In contrast to the case of $\mathbb{R}^n_+$, for the bounded Ω,

$$\mu = ||x|^{\sigma-s}u|^{2^*_\sigma} + \alpha_0\boldsymbol{\delta_0}(x),$$

and we should show that

$$\mathcal{M} \geqslant U + \mathcal{S}^{Sp}_{s,\sigma}(\Omega)\alpha_0^{\frac{2}{2^*_\sigma}}\boldsymbol{\delta_0}(x). \tag{44}$$

The estimate $\mathcal{M} \geqslant U$ coincides with (29). To show that $\mathcal{M}$ majorizes the second term of (44) we write the analogue of (30):

(45)

$$\int_\Omega \varphi^2_{2\epsilon}d\mathcal{M} = \lim_{k\to\infty}\int_0^{+\infty}\int_\Omega\Big[t^{1-2s}|\nabla_X\left[w_k(X)\varphi_{2\epsilon}\right]|^2 - 2t^{1-2s}|\nabla_x w_k\cdot\nabla_x\varphi_{2\epsilon}\cdot w_k\varphi_{2\epsilon}| + t^{1-2s}|w_k(X)\nabla_x\varphi_{2\epsilon}(x)|^2\Big]\,dX =: \widetilde{D}_1 - \widetilde{D}_2 + \widetilde{D}_3$$

with $\widetilde{D}_1 \geqslant \mathcal{S}^{Sp}_{s,\sigma}(\Omega)\alpha_0^{\frac{2}{2^*_\sigma}}$. The next step is the analogue of Lemma 3. Indeed, we have

$$\widetilde{D}_3 = \widetilde{a}_k + \widetilde{b}_k + \widetilde{c}_k := \left(\int_0^\delta + \int_\delta^{\frac{1}{\delta}} + \int_{\frac{1}{\delta}}^{+\infty}\right)\int_\Omega t^{1-2s}w_k^2(X)|\nabla_x\varphi_{2\epsilon}(x)|^2\,dX.$$

We can pass to the limit in $\widetilde{b}_k$, and the only remaining step is to obtain an analogue of (33). For a bounded Ω there is no explicit formula for the Green function, but we have the representation via Fourier series (see [31, (3.1)-(3.8)]):

$$w(X) = \sum_i d_i(t)\phi_i(x) \quad \text{with} \quad d_i(t) = t^s \frac{2^{1-s}}{\Gamma(s)} \lambda_i^{s/2} \langle u, \phi_i \rangle \mathcal{K}_s(\lambda_i^{1/2} t), \tag{46}$$

where $\mathcal{K}_s(\tau)$ is the modified Bessel function of the second kind; λ_i, ϕ_i were introduced in (5). The asymptotic behavior of $\mathcal{K}_s$ is (see, e.g., [31, (3.7)])

$$\mathcal{K}_s(\tau) \sim \Gamma(s) 2^{s-1} \tau^{-s} \text{ as } \tau \to 0; \quad \mathcal{K}_s(\tau) \sim \left(\frac{\pi}{2\tau}\right)^{\frac{1}{2}} e^{-\tau} \left(1 + O(\tau^{-1})\right) \text{ as } \tau \to \infty. \tag{47}$$

Since $\lambda_i \to \infty$, we can estimate

$$\int_\Omega w_k^2(X) |\nabla_x \varphi_{2\epsilon}(x)|^2 dx \leqslant \frac{C}{\epsilon^2} \sum_i \langle u_k, \phi_i \rangle^2 \cdot \begin{cases} 1 & \text{for} \quad t \in [0, \delta]; \\ t^{4s-4} & \text{for} \quad t \in \left[\frac{1}{\delta}, +\infty\right), \end{cases}$$

which gives

$$\widetilde{a}_k + \widetilde{c}_k \leqslant \frac{C}{\epsilon^2} \int_0^\infty \Big(t^{1-2s} \chi_{[0,\delta]}(t) + t^{2s-3} \chi_{\left[\frac{1}{\delta}, +\infty\right)}(t) \Big) \int_\Omega u_k^2 \, dX \leqslant C(\epsilon) \delta^{2-2s} \|u_k\|^2_{L_2(\Omega)}.$$

Further, repeating the argument from Section 5, we get (44). Similarly to (37) we have two alternatives: either $\alpha_0 = 0$ and the minimizer exists or $\alpha_0 = 1$ and $u \equiv 0$. We claim that in the second case the following inequality is fulfilled:

$$\mathcal{S}^{Sp}_{s,\sigma}(\Omega) \geqslant \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+). \tag{48}$$

Indeed, if $\{u_k\}$ is a minimizing sequence for (13), then $\{u_k \varphi_{2\epsilon}\}$ is a minimizing sequence too: the denominator of (13) converges to $\left[\alpha_0 \varphi_{2\epsilon}^{2^*_\sigma}(0)\right]^{\frac{2}{2^*_\sigma}} = \alpha_0^{\frac{2}{2^*_\sigma}}$, while the convergence of the numerator is controlled by (45) and Lemma 3 ($\widetilde{D}_2 = \widetilde{D}_3 = 0$):

$$\begin{aligned} \lim_{k\to\infty} \int_0^{+\infty} \int_\Omega t^{1-2s} \left| \nabla_X \left[w_k(X) \varphi_{2\epsilon} \right] \right|^2 dX = \int_\Omega \varphi_{2\epsilon}^2 d\mathcal{M} = \mathcal{S}^{Sp}_{s,\sigma}(\Omega) \alpha_0^{\frac{2}{2^*_\sigma}} \varphi_{2\epsilon}^2(0) \\ = \mathcal{S}^{Sp}_{s,\sigma}(\Omega) \alpha_0^{\frac{2}{2^*_\sigma}}. \end{aligned}$$

Therefore, we can assume that u_k is supported in $\mathbb{B}_{2\epsilon}$. Let $\Theta_1(x)$ be the coordinate transformation that flattens $\partial\Omega$ inside $\mathbb{B}_{r_0}$:

$$y \equiv (y', y_n) = \Theta_1(x) := (x', x_n - F(x')) = x - F(x') e_n.$$

The Jacobian of $\Theta_1(x)$ is equal to 1; thus

$$\begin{aligned} \mathcal{I}_{\sigma,\Omega}[u_k] &= \frac{C_s \mathcal{E}_s[w_k]}{\| |x|^{\sigma-s} u_k \|^2_{L_{2^*_\sigma}(\Omega)}} \\ &= \frac{\int_0^{+\infty} \int_{\mathbb{R}^n_+} z^{1-2s} |\nabla_Y w_k(y', y_n + F(y'), z)|^2 \, dY \cdot (1 + o_\epsilon(1))}{\int_{\mathbb{R}^n_+} \left| |y'|^2 + (y_n + F(y'))^2 \right|^{\frac{(\sigma-s)2^*_\sigma}{2}} \cdot u_k^{2^*_\sigma}(y', y_n + F(y')) \, dy}. \end{aligned}$$

Since $w_k(y', y_n + F(y'), z)$ is an admissible extension of $u_k(y', y_n + F(y'))$, we have

$$\mathcal{I}_{\sigma,\Omega}[u_k] \geqslant \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) \cdot (1 + o_\epsilon(1)),$$

which gives (48).

To complete the proof we use the assumptions (40)-(42) on $\partial\Omega$ to construct a function $\Phi_\varepsilon(x)$ such that $\mathcal{I}_{\sigma,\Omega}[\Phi_\varepsilon(x)] < \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)$. We define $\Theta_\varepsilon(x)$ and $\Theta_\varepsilon(X)$ as

$$\Theta_\varepsilon(x) := \varepsilon^{-1}\Theta_1(x), \quad \Theta_\varepsilon(X) := \big(\Theta_\varepsilon(x), \varepsilon^{-1}t\big) = \big(\varepsilon^{-1}(x - F(x')e_n), \varepsilon^{-1}t\big)\,. \tag{49}$$

The Jacobians of $\Theta_\varepsilon(x)$ and $\Theta_\varepsilon(X)$ are equal to ε^{-n} and ε^{-n-1}, respectively. Let $\delta \in (0, r_0)$. We define $\widetilde{\varphi}(x) := \varphi_\delta\left(\Theta_1(x)\right)$. Note that $\widetilde{\varphi}(\Theta_\varepsilon^{-1}(y))$ is radial:

$$\widetilde{\varphi}(\Theta_\varepsilon^{-1}(y)) = \widetilde{\varphi}(\varepsilon y', \varepsilon y_n + F(\varepsilon y')) = \varphi_\delta(\Theta_1(\Theta_\varepsilon^{-1}(y))) = \varphi_\delta(\varepsilon|y|).$$

Now we put

$$\Phi_\varepsilon(x) := \varepsilon^{-\frac{n-2s}{2}}\Phi(\Theta_\varepsilon(x))\widetilde{\varphi}(x); \quad w_\varepsilon(X) := \varepsilon^{-\frac{n-2s}{2}}\mathcal{W}(\Theta_\varepsilon(X))\widetilde{\varphi}(x)$$

(recall that $\Phi(y)$ is a minimizer of (13) in $\mathbb{R}^n_+$ and $\mathcal{W}(Y)$ is its Stinga–Torrea extension). Obviously, $w_\varepsilon(X)$ is an admissible extension of $\Phi_\varepsilon(x)$; therefore

$$\mathcal{I}_{\sigma,\Omega}[\Phi_\varepsilon(x)] = \frac{\langle(-\Delta)^s_{Sp}\Phi_\varepsilon, \Phi_\varepsilon\rangle}{\||x|^{\sigma-s}\Phi_\varepsilon(x)\|^2_{L_{2^*_\sigma}(\Omega)}} \leqslant \frac{\int_0^{+\infty}\int_\Omega t^{1-2s}|\nabla_X w_\varepsilon(X)|^2\,dX}{\||x|^{\sigma-s}\Phi_\varepsilon(x)\|^2_{L_{2^*_\sigma}(\Omega)}}. \tag{50}$$

In Sections 8 and 9 we derive the following estimates for the numerator and denominator on the right-hand side of (50):

$$\int_\Omega \frac{|\Phi_\varepsilon(x)|^{2^*_\sigma}}{|x|^{(s-\sigma)2^*_\sigma}}\,dx = 1 - \mathcal{A}_1(\varepsilon)\cdot(1 + o_\varepsilon(1) + o_\delta(1)); \tag{51}$$

(52)

$$\mathcal{E}_s\left[w_\varepsilon\right] = \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) + \mathcal{A}_2(\varepsilon)\cdot(1 + o_\varepsilon(1) + o_\delta(1)) - \frac{2\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)}{2^*_\sigma}\mathcal{A}_1(\varepsilon)\cdot(1 + o_\varepsilon(1)),$$

where $\mathcal{A}_1(\varepsilon), \mathcal{A}_2(\varepsilon) < 0$ and, for fixed δ and $\varepsilon \to 0$,

$$\mathcal{A}_1(\varepsilon) \sim c_1\varepsilon^{-1}f(\varepsilon), \quad \mathcal{A}_2(\varepsilon) \sim c_2\varepsilon^{-1}f(\varepsilon), \quad c_1, c_2 > 0.$$

Therefore, for sufficiently small δ and ε, we have

$$\begin{aligned}
&\mathcal{I}_{\sigma,\Omega}[\Phi_\varepsilon(x)] \\
&\quad \leqslant \frac{\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) + \mathcal{A}_2(\varepsilon)\cdot(1 + o_\varepsilon(1) + o_\delta(1)) - \frac{2\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)}{2^*_\sigma}\mathcal{A}_1(\varepsilon)\cdot(1 + o_\varepsilon(1))}{\Big(1 - \mathcal{A}_1(\varepsilon)\cdot(1 + o_\varepsilon(1) + o_\delta(1))\Big)^{\frac{2}{2^*_\sigma}}} \\
&\quad = \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) + \mathcal{A}_2(\varepsilon)\cdot(1 + o_\varepsilon(1) + o_\delta(1)) < \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+).
\end{aligned}$$

Thus, (48) is not fulfilled and a minimizer exists, which proves Theorem 3. □

7. Estimates for $\Phi(y)$ and $\mathcal{W}(Y)$

This section is devoted to the proof of Lemma 5. As a first step, we obtain the "rough" estimate for $\Phi(y)$ using the method from [30, Lemma 3.5] (see also [17, Sec. II.5]): it bounds Φ in terms of its modulus of continuity in Lebesgue space with the critical Sobolev exponent.

Let $0 < \tau < \|\Phi\|_{L_{2^*_s}(\mathbb{R}^n_+)}$. Then there exists the level $\lambda := \lambda\left(\Phi, \tau\right)$ such that

$$\|\Phi - \lambda\|_{L_{2^*_s}(\mathcal{Q}_\lambda)} = \tau, \quad \text{where} \quad \mathcal{Q}_\lambda := \{y \in \mathbb{R}^n_+ : \Phi(y) > \lambda\}.$$

Lemma 6. *There exists $\tau_*\left(n, s, \sigma\right)$ such that for any positive solution $\Phi(y)$ of* (14) *in $\mathbb{R}^n_+$:*

$$\sup\Phi \leqslant C\cdot\lambda(\Phi, \tau_*). \tag{53}$$

Proof. For any $\eta(Y) \in W_2^1(\mathbb{R}^n_+ \times \mathbb{R}_+, z^{1-2s})$, $\eta|_{y_n=0} = 0$ we have

$$\int_0^{+\infty} \int_{\mathbb{R}^n_+} z^{1-2s} \nabla_Y \mathcal{W}(Y) \cdot \nabla_Y \eta(Y)\, dY = \int_{\mathbb{R}^n_+} \frac{\Phi^{2^*_\sigma - 1}(y)}{|y|^{(s-\sigma)2^*_\sigma}} \eta(y,0)\, dy.$$

Taking $\eta(Y) := [\mathcal{W}(Y) - \lambda]_+$, we obtain

$$\begin{aligned}\mathcal{E}_{s,\lambda}[\mathcal{W}] := \int_{\{\mathcal{W}>\lambda\}} z^{1-2s} |\nabla_Y \mathcal{W}(Y)|^2\, dY &= \int_{\mathcal{Q}_\lambda} \frac{\Phi^{2^*_\sigma - 1}(y)}{|y|^{(s-\sigma)2^*_\sigma}} [\Phi(y) - \lambda]\, dy \\ &\leqslant \int_{\mathcal{Q}_\lambda} \frac{\Phi^{2^*_\sigma}(y)}{|y|^{(s-\sigma)2^*_\sigma}}\, dy.\end{aligned}$$

We estimate the integral from the right-hand side:

$$\begin{aligned}\|\Phi\|^{2^*_\sigma}_{L_{2^*_\sigma}\left(\mathcal{Q}_\lambda, |y|^{(\sigma-s)2^*_\sigma}\right)} &\leqslant \left(\|\Phi - \lambda\|_{L_{2^*_\sigma}\left(\mathcal{Q}_\lambda, |y|^{(\sigma-s)2^*_\sigma}\right)} + \lambda \|1\|_{L_{2^*_\sigma}\left(\mathcal{Q}_\lambda, |y|^{(\sigma-s)2^*_\sigma}\right)} \right)^{2^*_\sigma} \\ &\leqslant 2^{2^*_\sigma} \left(\|\Phi - \lambda\|^{2^*_\sigma}_{L_{2^*_\sigma}} + \lambda^{2^*_\sigma} \int_{\mathcal{Q}_\lambda} \frac{1}{|y|^{(s-\sigma)2^*_\sigma}}\, dy \right) \\ &\overset{*}{\leqslant} 2^{2^*_\sigma} \|\Phi - \lambda\|^{2^*_\sigma}_{L_{2^*_\sigma}\left(\mathcal{Q}_\lambda, |y|^{(\sigma-s)2^*_\sigma}\right)} + C_1 \lambda^{2^*_\sigma} |\mathcal{Q}_\lambda|^{\frac{n-2s}{n-2\sigma}}.\end{aligned}$$

The inequality (*) follows from the Schwarz symmetrization. Recall that $\tau \equiv \|\Phi - \lambda\|_{L_{2^*_s}(\mathcal{Q}_\lambda)}$; using the Hölder inequality we get

$$\begin{aligned}\|\Phi - \lambda\|^{2^*_\sigma}_{L_{2^*_\sigma}\left(\mathcal{Q}_\lambda, |y|^{(\sigma-s)2^*_\sigma}\right)} &\leqslant \|\Phi - \lambda\|^{\frac{2n(s-\sigma)}{(n-2\sigma)s}}_{L_2(\mathcal{Q}_\lambda, |y|^{-2s})} \cdot \|\Phi - \lambda\|^{2 - \frac{2n(s-\sigma)}{(n-2\sigma)s} + 2^*_\sigma - 2}_{L_{2^*_s}(\mathcal{Q}_\lambda)} \\ &= \|\Phi - \lambda\|^{\frac{2n(s-\sigma)}{(n-2\sigma)s}}_{L_2(\mathcal{Q}_\lambda, |y|^{-2s})} \cdot \|\Phi - \lambda\|^{2 - \frac{2n(s-\sigma)}{(n-2\sigma)s}}_{L_{2^*_s}(\mathcal{Q}_\lambda)} \cdot \tau^{2^*_\sigma - 2}.\end{aligned}$$

Due to the fractional Hardy and Sobolev inequalities

$$\begin{aligned}\|\Phi - \lambda\|^{\frac{2n(s-\sigma)}{(n-2\sigma)s}}_{L_2(\mathcal{Q}_\lambda, |y|^{-2s})} \cdot \|\Phi - \lambda\|^{2 - \frac{2n(s-\sigma)}{(n-2\sigma)s}}_{L_{2^*_s}(\mathcal{Q}_\lambda)} &\leqslant C_2 \langle (-\Delta)^s_{\mathcal{Q}_\lambda, Sp} [\Phi - \lambda]_+, [\Phi - \lambda]_+ \rangle \\ &\overset{**}{\leqslant} C_2 \mathcal{E}_{s,\lambda}[\mathcal{W}].\end{aligned}$$

The inequality (**) follows from the fact that $\eta(Y)$ is an admissible extension of $[\Phi - \lambda]_+$. To sum up,

$$\mathcal{E}_{s,\lambda}[\mathcal{W}] \leqslant 2^*_\sigma C_2 \mathcal{E}_{s,\lambda}[\mathcal{W}]\, \tau^{2^*_\sigma - 2} + C_1 \lambda^{2^*_\sigma} |\mathcal{Q}_\lambda|^{\frac{n-2s}{n-2\sigma}}.$$

Suppose that τ_* satisfies $2^{2^*_\sigma} C_2 \tau_*^{2^*_\sigma - 2} \leqslant \frac{1}{2}$. For all $\lambda > \lambda(\Phi, \tau_*)$ we have

$$C_3 \|\Phi - \lambda\|^2_{L_{2^*_s}(\mathcal{Q}_\lambda)} \leqslant \mathcal{E}_{s,\lambda}[\mathcal{W}] \leqslant 2C_1 \lambda^{2^*_\sigma} |\mathcal{Q}_\lambda|^{\frac{n-2s}{n-2\sigma}}. \tag{54}$$

From (54) we obtain

$$\mathfrak{g}(\lambda) := \int_{\mathcal{Q}_\lambda} [\Phi(y) - \lambda]\, dy \leqslant \|\Phi - \lambda\|_{L_{2^*_s}(\mathcal{Q}_\lambda)} \cdot |\mathcal{Q}_\lambda|^{\frac{n+2s}{2n}} \leqslant C_4 \lambda^{\frac{n}{n-2\sigma}} |\mathcal{Q}_\lambda|^{1 + \frac{\sigma(n-2s)}{n(n-2\sigma)}}. \tag{55}$$

Using the layer cake representation for the Lebesgue integral

$$\mathfrak{g}(\lambda)=\int_{\mathcal{Q}_\lambda}\int_\lambda^\infty \chi_{\{\theta<\Phi(y)\}}\,d\theta\,dy=\int_\lambda^\infty |\mathcal{Q}_\theta|\,d\theta,$$

we get $\mathfrak{g}'(\lambda)=-|\mathcal{Q}_\lambda|$ for a.e. λ. Thus (55) takes the form

$$-\mathfrak{g}'(\lambda)\,[\mathfrak{g}(\lambda)]^{-\frac{n(n-2\sigma)}{n^2-n\sigma-2\sigma s}}\geqslant C_5\lambda^{-\frac{n^2}{n^2-n\sigma-2\sigma s}}.$$

By integrating over the segment $[\lambda,\sup\Phi]$ we get

$$-\mathfrak{g}(\lambda)^{\frac{n\sigma-2\sigma s}{n^2-n\sigma-2\sigma s}}\leqslant C_6\left[(\sup\Phi)^{-\frac{n\sigma+2\sigma s}{n^2-n\sigma-2\sigma s}}-\lambda^{-\frac{n\sigma+2\sigma s}{n^2-n\sigma-2\sigma s}}\right];$$

$$(\sup\Phi)^{-\frac{n\sigma+2\sigma s}{n^2-n\sigma-2\sigma s}}\geqslant\lambda^{-\frac{n\sigma+2\sigma s}{n^2-n\sigma-2\sigma s}}-C_6^{-1}\mathfrak{g}(\lambda)^{\frac{n\sigma-2\sigma s}{n^2-n\sigma-2\sigma s}}.$$

Using (55) for $\tau_*\leqslant\left(\frac{C_6}{2}\right)^{\frac{n^2-n\sigma-2\sigma s}{n\sigma-2\sigma s}}\cdot\left[\mathcal{S}_{s,s}^{-1}\cdot\mathcal{S}_{s,\sigma}^{Sp}(\mathbb{R}^n_+)\right]^{-\frac{n+2s}{2(n-2s)}}$ we obtain

$$\begin{aligned}\mathfrak{g}(\lambda)\lambda^{\frac{n+2s}{n-2s}}&\leqslant\|\Phi-\lambda\|_{L_{2_s^*}(\mathcal{Q}_\lambda)}\left(|\mathcal{Q}_\lambda|\lambda^{2_s^*}\right)^{\frac{n+2s}{2n}}\leqslant\tau_*\|\Phi\|_{L_{2_s^*}}^{2_s^*-1}\\&\leqslant\tau_*\left[\mathcal{S}_{s,s}^{-1}\mathcal{E}_s[\mathcal{W}]\right]^{\frac{n+2s}{2(n-2s)}}=\tau_*\left[\mathcal{S}_{s,s}^{-1}\cdot\mathcal{S}_{s,\sigma}^{Sp}(\mathbb{R}^n_+)\right]^{\frac{n+2s}{2(n-2s)}}\\&\leqslant\left(\frac{C_6}{2}\right)^{\frac{n^2-n\sigma-2\sigma s}{n\sigma-2\sigma s}},\end{aligned}\tag{56}$$

which gives the required inequality (53):

$$(\sup\Phi)^{-\frac{n\sigma+2\sigma s}{n^2-n\sigma-2\sigma s}}\geqslant\frac{1}{2}\lambda^{-\frac{n\sigma+2\sigma s}{n^2-n\sigma-2\sigma s}}.$$ □

Corollary 1. *Any minimizer $\Phi(y)$ admits the estimate (τ_* was introduced in Lemma 6):*

$$\Phi(y)\leqslant\frac{C(n,s,\sigma,\lambda(\Phi,\tau_*),\lambda(\Phi^*,\tau_*))}{(1+|y|)^{n-2s}}.\tag{57}$$

Proof. For $|y|\leqslant 1$ the estimate (57) coincides with (53). For $|y|\geqslant 1$ the estimate (57) can be obtained via the s-Kelvin transform (15):

$$\Phi(y)\leqslant\frac{1}{|y|^{n-2s}}\cdot\sup\Phi^*\left(\frac{y}{|y|^2}\right)\leqslant\frac{C(n,s,\sigma,\lambda(\Phi^*,\tau_*))}{|y|^{n-2s}}.$$

□

Proof of Lemma 5. The estimate for $\Phi(y)$ in (38) follows from the estimate for $\mathcal{W}(Y)$ due to $\Phi(y)=\mathcal{W}(y,0)$. Moreover, the s-Kelvin transform argument shows that it suffices to prove (38) for $|Y|\leqslant 1$ only. Using the Green function (19), we can write

$$\mathcal{W}(Y)=A_1+A_2+A_3:=\left(\int_{|\xi|>2}+\int_{\substack{|\xi|\leqslant 2\\ |y-\xi|>\frac{y_n}{2}}}+\int_{\substack{|\xi|\leqslant 2\\ |y-\xi|\leqslant\frac{y_n}{2}}}\right)G_s(Y,\xi)\,\frac{\Phi^{2_\sigma^*-1}(\xi)}{|\xi|^{(s-\sigma)2_\sigma^*}}\,d\xi.$$

To estimate A_1, we use (57) and (24) with $\mathfrak{b}=1$:

$$\begin{aligned}A_1&\leqslant Cy_n\int_{|\xi|>2}|\xi|^{(2_\sigma^*-1)(2s-n)}|\xi|^{(\sigma-s)2_\sigma^*}\frac{\xi_n}{|\xi|^{n-2s+2}}\,d\xi\\&\leqslant Cy_n\int_{|\xi|>2}|\xi|^{-\left(\frac{n^2-2ns}{n-2\sigma}+n+1\right)}\,d\xi\leqslant Cy_n.\end{aligned}\tag{58}$$

The estimates on A_2 and A_3 are obtained iteratively. Recall that we have fixed the minimizer $\Phi(y)$. Let the following a priori estimate with $\mathfrak{p} \in [0,1)$ be fulfilled (for $\mathfrak{p} = 0$ it was proved in Lemma 6):

$$\Phi(y) \leqslant C y_n^{\mathfrak{p}}. \tag{59}$$

We claim that (59) implies that

$$\mathcal{W}(Y) \leqslant C y_n^{\mathfrak{p}_*} \quad \text{and} \quad \Phi(y) \leqslant C y_n^{\mathfrak{p}_*}, \tag{60}$$

with $\mathfrak{p}_* := \min(\mathfrak{q} + \mathfrak{p}, 1)$ and

$$\mathfrak{q} := \frac{\sigma(n-2s)}{n-2\sigma} = s - \frac{(s-\sigma)2^*_\sigma}{2} \in (0,s).$$

Indeed, to estimate A_2 we notice that on the integration set one has

$$\xi_n \leqslant |\xi - y| + y_n \leqslant 3|\xi - y|.$$

Therefore the inequalities $\xi_n < |\xi|$, (57), and (24) with $\mathfrak{b} = 1$ give us

$$\begin{aligned} A_2 &\leqslant C \int_{\substack{|\xi|\leqslant 2 \\ |y-\xi|>\frac{y_n}{2}}} |\xi|^{(\sigma-s)2^*_\sigma} \frac{y_n \xi_n^{1+(2^*_\sigma-1)\mathfrak{p}}}{|y-\xi|^{n-2s+2}}\, d\xi \\ &\leqslant C y_n^{\mathfrak{p}_*} \int_{|\xi|\leqslant 2} |\xi|^{2(\mathfrak{q}-s)} |y-\xi|^{-n+2s+(2^*_\sigma-1)\mathfrak{p}-\mathfrak{p}_*}\, d\xi, \end{aligned} \tag{61}$$

which in particular implies that A_2 is finite, because both of the exponents are negative and their sum is greater than $-n$.

To estimate A_3, we notice that on the integration set one has

$$|\xi| \geqslant |y| - |y-\xi| \geqslant |y| - \frac{y_n}{2} \geqslant \frac{y_n}{2}; \quad \xi_n \leqslant |y_n - \xi_n| + y_n \leqslant \frac{3y_n}{2}.$$

Therefore (24) with $\mathfrak{b} = 0$ gives us

$$\begin{aligned} A_3 &\leqslant \int_{|y-\xi|\leqslant \frac{y_n}{2}} \frac{C \xi_n^{(2^*_\sigma-1)\mathfrak{p}}}{|\xi|^{2s-2\mathfrak{q}} |y-\xi|^{n-2s}}\, d\xi \\ &\leqslant C y_n^{(2^*_\sigma-1)\mathfrak{p}-2s+2\mathfrak{q}} \int_{|y-\xi|\leqslant \frac{y_n}{2}} \frac{1}{|y-\xi|^{n-2s}}\, d\xi \leqslant C y_n^{\mathfrak{p}_*}. \end{aligned} \tag{62}$$

Putting (58), (61), and (62) together, we obtain (60); i.e. we have increased the exponent in (59) by at least $\min(\mathfrak{q}, 1-\mathfrak{p})$. Iterating this process, we get (60) with $\mathfrak{p}_* = 1$. The estimate (38) is completely proved.

To prove (39) we have to derive estimates at the origin and at infinity separately because $\mathcal{V}(y)$ is not invariant under the s-Kelvin transform. For $|y| \leqslant 1$, we write the integral representation for $\nabla_Y \mathcal{W}(Y)$ as

$$\nabla_Y \mathcal{W}(Y) = \left(\int_{|\xi|\geqslant 2} + \int_{|\xi|<2} \right) \frac{\Phi^{2^*_\sigma-1}(\xi)}{|\xi|^{(s-\sigma)2^*_\sigma}} \nabla_Y G_s(Y,\xi)\, d\xi =: A_4 + A_5.$$

Obviously,

$$\mathcal{V}(y) \leqslant 2\int_0^{+\infty} z^{1-2s} A_4^2(Y)\,dz + 2\int_0^{2} z^{1-2s} A_5^2(Y)\,dz + 2\int_2^{+\infty} z^{1-2s} A_5^2(Y)\,dz. \tag{63}$$

We estimate A_4 using (25) and (38):

$$A_4 \leqslant \int_{|\xi|\geqslant 2} \frac{\xi_n^{2^*_\sigma-1}}{|\xi|^{(s-\sigma)2^*_\sigma+(2^*_\sigma-1)(n-2s+2)}\left(|y-\xi|^2+z^2\right)^{\frac{n-2s+1}{2}}}\,d\xi.$$

Therefore, taking into account $|y-\xi| \geqslant |\xi| - |y| \geqslant 1$ we get

$$\int_0^{+\infty} z^{1-2s} A_4^2(Y)\,dz$$
$$\leqslant C\int_0^{+\infty} \frac{z^{1-2s}}{(1+z^2)^{n-2s+1}}\,dz \cdot \left(\int_2^{+\infty} \frac{r^{2^*_\sigma-1} r^{n-1}}{r^{(s-\sigma)2^*_\sigma+(2^*_\sigma-1)(n-2s+2)}}\,dr\right)^2 \leqslant C.$$

The convergence of the last integral follows from the equality

$$2^*_\sigma - 1 + n - 1 - (s-\sigma)2^*_\sigma - (2^*_\sigma - 1)(n-2s+2) = -\frac{2\sigma(n-2s+2)}{n-2\sigma} - 2.$$

The estimate of A_5 also follows from (25) and (38):

$$A_5 \leqslant \int_{|\xi|<2} \frac{\xi_n^{2^*_\sigma-1}}{|\xi|^{(s-\sigma)2^*_\sigma}\left(|y-\xi|^2+z^2\right)^{\frac{n-2s+1}{2}}}\,d\xi.$$

Using this inequality, we estimate the second term in (63):

$$\int_0^{2} z^{1-2s} A_5^2(Y)\,dz$$
$$\leqslant C\int_0^{2} z^{-1+\min(s,1-s)}\,dz \cdot \left(\int_{|\xi|<2} \frac{\xi_n^{2^*_\sigma-1}}{|\xi|^{(s-\sigma)2^*_\sigma}|y-\xi|^{n-s+\frac{\min(s,1-s)}{2}}}\,d\xi\right)^2 \leqslant C.$$

The convergence of the last integral follows from the inequality

$$2^*_\sigma - 2 - (s-\sigma)2^*_\sigma + s - \frac{\min(s,1-s)}{2} = \frac{2\sigma(n-2s+2)}{n-2\sigma} - s - \frac{\min(s,1-s)}{2} > -1.$$

Finally, the third term in (63) can be estimated as

$$\int_2^{+\infty} z^{1-2s} A_5^2(Y)\,dz \leqslant C\int_2^{+\infty} \frac{z^{1-2s}}{z^{2n-4s+2}}\,dz \cdot \left(\int_0^{2} \frac{r^{2^*_\sigma-1} r^{n-1}}{r^{(s-\sigma)2^*_\sigma}}\,dr\right)^2 \leqslant C,$$

and (39) is proved for $|y| \leqslant 1$.

For $|y| > 1$, we write the integral representation for $\nabla_Y \mathcal{W}(Y)$ as

$$\nabla_Y \mathcal{W}(Y) = \left(\int_{|y-\xi|<\frac{|y|}{10}} + \int_{|y-\xi|\geqslant\frac{|y|}{10}} \right) \frac{\Phi^{2^*_\sigma - 1}(\xi)}{|\xi|^{(s-\sigma)2^*_\sigma}} \nabla_Y G_s(Y,\xi)\, d\xi =: A_6 + A_7.$$

Then $\mathcal{V}(y)$ can be estimated with an obvious inequality:

$$\mathcal{V}(y) \leqslant 2 \int_0^{+\infty} z^{1-2s} A_6^2(Y)\, dz + 2 \int_0^{+\infty} z^{1-2s} A_7^2(Y)\, dz. \tag{64}$$

We estimate A_6 using (25), (38), and $|\xi| \geqslant \frac{9|y|}{10} \geqslant \frac{9}{10}$:

$$\begin{aligned} A_6 &\leqslant C \int_{|y-\xi|<\frac{|y|}{10}} \frac{|\xi|^{(\sigma-s)2^*_\sigma - (2^*_\sigma-1)(n-2s+1)}}{\left(|y-\xi|^2+z^2\right)^{\frac{n-2s+1}{2}}}\, d\xi \\ &\leqslant \frac{C}{|y|^{(s-\sigma)2^*_\sigma + (2^*_\sigma-1)(n-2s+1)}} \int_0^{\frac{|y|}{10}} \frac{r^{n-1}}{\left(r^2+z^2\right)^{\frac{n-2s+1}{2}}}\, dr. \end{aligned}$$

By changing the variable we see that

$$\begin{aligned} &\int_0^{+\infty} z^{1-2s} \left(\int_0^{\frac{|y|}{10}} \frac{r^{n-1}}{\left(r^2+z^2\right)^{\frac{n-2s+1}{2}}}\, dr \right)^2 dz \\ &= \frac{|y|^{2s}}{10^{2s}} \int_0^{+\infty} z^{1-2s} \left(\int_0^{1} \frac{r^{n-1}}{\left(r^2+z^2\right)^{\frac{n-2s+1}{2}}}\, dr \right)^2 dz, \end{aligned}$$

which gives the estimate of the first term in (64):

$$\int_0^{+\infty} z^{1-2s} A_6^2(Y)\, dz \leqslant \frac{C|y|^{2s}}{|y|^{2((s-\sigma)2^*_\sigma + (2^*_\sigma-1)(n-2s+1))}} = \frac{C}{|y|^{2n-2s+2+4\sigma\frac{n-2s+2}{n-2\sigma}}}.$$

Finally, we estimate A_7 using (25) and (38):

$$\begin{aligned} A_7 &\leqslant \int_{|y-\xi|\geqslant\frac{|y|}{10}} \frac{C\Phi^{2^*_\sigma-1}(\xi)\xi_n}{|\xi|^{(s-\sigma)2^*_\sigma}\left(|y-\xi|^2+z^2\right)^{\frac{n-2s+2}{2}}} \left(\frac{y_n}{\sqrt{|y-\xi|^2+z^2}} + 1 \right) d\xi \\ &\leqslant \frac{C}{\left(|y|^2+z^2\right)^{\frac{n-2s+2}{2}}} \cdot \int_{|y-\xi|\geqslant\frac{|y|}{10}} \frac{\xi_n^{2^*_\sigma}}{|\xi|^{(s-\sigma)2^*_\sigma}\left(1+|\xi|^{(2^*_\sigma-1)(n-2s+2)}\right)}\, d\xi. \end{aligned}$$

Convergence of the last integral follows from the inequality,

$$2^*_\sigma - (s-\sigma)2^*_\sigma - (2^*_\sigma - 1)(n-2s+2) = -\frac{n^2-4s\sigma+4\sigma}{n-2\sigma} = -n-2\sigma\frac{n-2s+2}{n-2\sigma} < -n.$$

This gives the estimate of the second term in (64),

$$\int_0^{+\infty} z^{1-2s} A_7^2(Y)\, dz \leqslant C \int_0^{+\infty} \frac{z^{1-2s}}{(|y|^2+z^2)^{n-2s+2}}\, dz \leqslant \frac{C}{|y|^{2n-2s+2}},$$

and the estimate (39) is completely proved. □

8. Estimate on the denominator and derivation of (51)

To get (51) we modify the calculations from [6, Sec. 4]. We use the change of variables (49) and obtain the following equality by the Taylor formula:

$$\begin{aligned}
&\int_\Omega \frac{|\Phi_\varepsilon(x)|^{2^*_\sigma}}{|x|^{(s-\sigma)2^*_\sigma}}\, dx \\
&= \int_{\mathbb{R}^n_+} \frac{|\Phi(y)|^{2^*_\sigma}}{|y+\varepsilon^{-1}F(\varepsilon y')e_n|^{(s-\sigma)2^*_\sigma}} \widetilde{\varphi}^{2^*_\sigma}(\Theta_\varepsilon^{-1}(y))\, dy \\
&= \int_{\mathbb{R}^n_+} \frac{|\Phi(y)|^{2^*_\sigma}}{|y|^{(s-\sigma)2^*_\sigma}} \varphi_\delta^{2^*_\sigma}(\varepsilon y) \cdot \left(1 - \frac{(s-\sigma)2^*_\sigma}{\varepsilon} F(\varepsilon y') \frac{y_n}{|y|^2} + \frac{F^2(\varepsilon y')}{\varepsilon^2 |y|^2} \cdot O_\delta(1)\right) dy \\
&= \int_{\mathbb{R}^n_+} \frac{|\Phi(y)|^{2^*_\sigma}}{|y|^{(s-\sigma)2^*_\sigma}}\, dy - \int_{\mathbb{R}^n_+} \frac{|\Phi(y)|^{2^*_\sigma}}{|y|^{(s-\sigma)2^*_\sigma}} \left(1-\varphi_\delta^{2^*_\sigma}(\varepsilon y)\right) dy \\
&\quad - \int_{\mathbb{R}^n_+} \frac{(s-\sigma)2^*_\sigma |\Phi(y)|^{2^*_\sigma} \varphi_\delta^{2^*_\sigma}(\varepsilon y) y_n}{\varepsilon |y|^{(s-\sigma)2^*_\sigma+2}} F(\varepsilon y')\, dy \\
&+ O_\delta(1) \int_{\mathbb{R}^n_+} \frac{|\Phi(y)|^{2^*_\sigma}}{|y|^{(s-\sigma)2^*_\sigma}} \varphi_\delta^{2^*_\sigma}(\varepsilon y) \frac{F^2(\varepsilon y')}{\varepsilon^2|y|^2}\, dy =: I_1 - I_2 - I_3 + I_4.
\end{aligned}$$

Lemma 7. *The following relations hold:*

(1) $I_1 = 1$ *and* $I_2 \leqslant C\left(\frac{\varepsilon}{\delta}\right)^{\frac{n(n-2s+2)}{n-2\sigma}}$;
(2)

$$\lim_{\varepsilon\to 0} \varepsilon \frac{I_3}{f(\varepsilon)} = C \int_0^{+\infty} \tau^{\alpha+n} \int_0^{+\infty} \frac{|\Phi(\tau,\tau\varsigma)|^{2^*_\sigma} \varsigma d\varsigma}{|\tau^2+\tau^2\varsigma^2|^{\frac{(s-\sigma)2^*_\sigma+2}{2}}}\, d\tau < +\infty; \tag{65}$$

(3) $\lim_{\varepsilon\to 0} \left|\varepsilon \frac{I_4}{f(\varepsilon)}\right| = o_\delta(1)$.

Proof. (1) The equality $I_1 = 1$ is just a normalizing condition for $\Phi(y)$. Further, (38) gives

$$\begin{aligned}
I_2 &\equiv \int_{\mathbb{R}^n_+} \frac{|\Phi(y)|^{2^*_\sigma}}{|y|^{(s-\sigma)2^*_\sigma}} \left(1-\varphi_\delta^{2^*_\sigma}(\varepsilon y)\right) dy \leqslant C \int_{\frac{\delta}{2\varepsilon}}^{+\infty} r^{n-1-2^*_\sigma(n-s-\sigma+1)}\, dr \\
&= C\left(\frac{\varepsilon}{\delta}\right)^{\frac{n(n-2s+2)}{n-2\sigma}}.
\end{aligned}$$

(2) We note that

$$\begin{aligned}\frac{\varepsilon I_3}{f(\varepsilon)} &= \frac{(s-\sigma)2^*_\sigma}{f(\varepsilon)}\int_{\mathbb{R}^n_+}\varphi_\delta^{2^*_\sigma}(\varepsilon y)\frac{|\Phi(y)|^{2^*_\sigma}y_n}{|y|^{(s-\sigma)2^*_\sigma+2}}F(\varepsilon y')\,dy\\ &= \frac{C}{f(\varepsilon)}\int_0^{+\infty}\int_0^{+\infty}\varphi_\delta^{2^*_\sigma}(\varepsilon\sqrt{\tau^2+y_n^2})\frac{|\Phi(\tau,y_n)|^{2^*_\sigma}y_n}{(\tau^2+y_n^2)^{\frac{(s-\sigma)2^*_\sigma+2}{2}}}\int_{\mathbb{S}^{n-2}_\tau}F(\varepsilon y')\,d\mathbb{S}^{n-2}_\tau(y')\,dy_n\,d\tau\\ &= C\int_0^{+\infty}\tau^n\frac{f(\varepsilon\tau)}{f(\varepsilon)}\int_0^{+\infty}\varphi_\delta^{2^*_\sigma}(\varepsilon\sqrt{\tau^2+\tau^2\varsigma^2})\frac{|\Phi(\tau,\tau\varsigma)|^{2^*_\sigma}\varsigma\,d\varsigma}{(\tau^2+\tau^2\varsigma^2)^{\frac{(s-\sigma)2^*_\sigma+2}{2}}}\,d\tau\\ &= C\int_0^{+\infty}\tau^{\alpha+n}\frac{\psi(\varepsilon\tau)}{\psi(\varepsilon)}\int_0^{+\infty}\varphi_\delta^{2^*_\sigma}(\varepsilon\sqrt{\tau^2+\tau^2\varsigma^2})\frac{|\Phi(\tau,\tau\varsigma)|^{2^*_\sigma}\varsigma\,d\varsigma}{(\tau^2+\tau^2\varsigma^2)^{\frac{(s-\sigma)2^*_\sigma+2}{2}}}\,d\tau\\ &=: C\int_0^{+\infty}P_\varepsilon(\tau)\,d\tau.\end{aligned}$$

The pointwise limit of $P_\varepsilon(\tau)$ as $\varepsilon\to 0$ coincides with the integrand in the right-hand side of (65). To get the final result we use the Lebesgue dominated convergence theorem. To construct a summable majorant for $P_\varepsilon(\tau)$ we notice that $\psi(\tau)$ is an SVF and therefore $\psi(\tau)\tau^\beta$ increases and $\psi(\tau)\tau^{-\beta}$ decreases in the neighbourhood of the origin for any $\beta>0$; see [29, Sec 1.5, (1)-(2)]. This implies that

$$\begin{aligned}\chi_{[0,\frac{\delta}{\varepsilon}]}(\tau)\frac{\psi(\varepsilon\tau)}{\psi(\varepsilon)} &= \frac{\psi(\varepsilon\tau)(\varepsilon\tau)^\beta}{\psi(\varepsilon)(\varepsilon)^\beta}\chi_{[0,1]}(\tau)\tau^{-\beta}+\frac{\psi(\varepsilon\tau)(\varepsilon\tau)^{-\beta}}{\psi(\varepsilon)(\varepsilon)^{-\beta}}\chi_{[1,\frac{\delta}{\varepsilon}]}(\tau)\tau^\beta\\ &\leqslant C(\delta)\left(\chi_{[0,1]}(\tau)\tau^{-\beta}+\chi_{[1,+\infty)}(\tau)\tau^\beta\right).\end{aligned}\tag{66}$$

Thus,

$$P_\varepsilon(\tau)\leqslant C(\delta)\left(\chi_{[0,1]}(\tau)\tau^{\alpha+n-\beta}+\chi_{[1,+\infty)}(\tau)\tau^{\alpha+n+\beta}\right)\int_0^{+\infty}\frac{|\Phi(\tau,\tau\varsigma)|^{2^*_\sigma}\varsigma}{(\tau^2+\tau^2\varsigma^2)^{\frac{(s-\sigma)2^*_\sigma+2}{2}}}\,d\varsigma.$$

By (38), for $\tau\in[0,1]$ we have

$$\begin{aligned}\int_0^{+\infty}\frac{(\tau^2+\tau^2\varsigma^2)^{\frac{(1-s+\sigma)2^*_\sigma-2}{2}}\varsigma}{1+(\tau^2+\tau^2\varsigma^2)^{\frac{(n-2s+2)2^*_\sigma}{2}}}\,d\varsigma &= \frac{1}{2\tau^2}\int_{\tau^2}^{+\infty}\frac{r^{\frac{(1-s+\sigma)2^*_\sigma-2}{2}}}{1+r^{\frac{(n-2s+2)2^*_\sigma}{2}}}\,dr\\ &\leqslant \frac{1}{2\tau^2}\int_0^{+\infty}\frac{r^{\frac{(1-s+\sigma)2^*_\sigma-2}{2}}}{1+r^{\frac{(n-2s+2)2^*_\sigma}{2}}}\,dr,\end{aligned}$$

while for $\tau>1$ we have

$$\int_0^{+\infty}\frac{(\tau^2+\tau^2\varsigma^2)^{\frac{(1-s+\sigma)2^*_\sigma-2}{2}}\varsigma}{1+(\tau^2+\tau^2\varsigma^2)^{\frac{(n-2s+2)2^*_\sigma}{2}}}\,d\varsigma\leqslant\tau^{-(n-s-\sigma+1)2^*_\sigma-2}\int_0^{+\infty}\frac{\varsigma}{(1+\varsigma^2)^{\frac{(n-s-\sigma+1)2^*_\sigma+2}{2}}}\,d\varsigma.$$

So, choosing sufficiently small β, we get an estimate

$$P_\varepsilon(\tau)\leqslant C\left(\chi_{[0,1]}(\tau)\tau^{\alpha+n-2-\beta}+\chi_{[1,+\infty)}(\tau)\tau^{\alpha+n+\beta-(n-s-\sigma+1)2^*_\sigma-2}\right)$$

with the summable majorant in the right-hand side (recall that $\alpha<n-2s+3$):

$$\alpha+n+\beta-(n-s-\sigma+1)2^*_\sigma-2<-1+\beta-\frac{2\sigma(n-2s+2)}{n-2\sigma}<-1.$$

(3) Using (43), we obtain

$$\left|\varepsilon\frac{I_4}{f(\varepsilon)}\right| \leqslant O_\delta(1)\int_0^{\frac{\delta}{\varepsilon}} \frac{\tau^{n-2}f_1(\varepsilon\tau)}{\varepsilon|f(\varepsilon)|}\int_0^{\sqrt{\frac{\delta^2}{\varepsilon^2}-\tau^2}} \frac{|\Phi(\tau,y_n)|^{2^*_\sigma}\,dy_n}{(\tau^2+y_n^2)^{\frac{(s-\sigma)2^*_\sigma+2}{2}}}\,d\tau$$
$$\leqslant o_\delta(1)\int_0^{\frac{\delta}{\varepsilon}} \frac{\tau^n f(\varepsilon\tau)}{f(\varepsilon)}\int_0^{\sqrt{\frac{\delta^2}{\varepsilon^2\tau^2}-1}} \frac{|\Phi(\tau,\tau\varsigma)|^{2^*_\sigma}\,d\varsigma}{(\tau^2+\tau^2\varsigma^2)^{\frac{(s-\sigma)2^*_\sigma+2}{2}}}\,d\tau.$$

Similar to the previous estimate, the integral in the right-hand side has the finite limit as $\varepsilon\to 0$, which completes the proof. □

To get (51) we put $\mathcal{A}_1(\varepsilon) := I_3$; estimates $I_4 = o_\delta(1)\mathcal{A}_1(\varepsilon)$ and $I_2 = o_\varepsilon(1)\mathcal{A}_1(\varepsilon)$ follow from Lemma 7 and the inequality

$$I_2 \leqslant C(\delta)\cdot\varepsilon^{\frac{n(n-2s+2)}{n-2\sigma}} = o_\varepsilon(1)\cdot\varepsilon^{\alpha-1} \leqslant o_\varepsilon(1)\cdot\varepsilon^{-1}f(\varepsilon) = o_\varepsilon(1)\cdot\mathcal{A}_1(\varepsilon).$$

9. Estimate on the numerator and derivation of (52)

For brevity, we denote $\mathfrak{y} := \frac{n-2s}{2}$. For $i\in\{1,\dots,n-1\}$ we have

$$\begin{pmatrix}\partial_t w_\varepsilon(X)\\ \partial_{x_i} w_\varepsilon(X)\\ \partial_{x_n} w_\varepsilon(X)\end{pmatrix}$$
$$= \begin{pmatrix}\varepsilon^{-\mathfrak{y}-1}\mathcal{W}_z(\Theta_\varepsilon(X))\widetilde{\varphi}(x)\\ \varepsilon^{-\mathfrak{y}-1}[\mathcal{W}_{y_i}(\Theta_\varepsilon(X))-\mathcal{W}_{y_n}(\Theta_\varepsilon(X))F_{x_i}(x')]\widetilde{\varphi}(x)+\varepsilon^{-\mathfrak{y}}\mathcal{W}(\Theta_\varepsilon(X))\widetilde{\varphi}_{x_i}(x)\\ \varepsilon^{-\mathfrak{y}-1}\mathcal{W}_{y_n}(\Theta_\varepsilon(X))\widetilde{\varphi}(x)+\varepsilon^{-\mathfrak{y}}\mathcal{W}(\Theta_\varepsilon(X))\widetilde{\varphi}_{x_n}(x)\end{pmatrix}.$$

Using these formulae we get the representation for the energy

$$\begin{aligned}
&\mathcal{E}_s[w_\varepsilon]\\
&= \int_0^{+\infty} t^{1-2s}\int_\Omega\Bigg(\sum_{i=1}^{n-1}\Big[\varepsilon^{-2\mathfrak{y}-2}\widetilde{\varphi}^2(x)\mathcal{W}^2_{y_i}(\Theta_\varepsilon(X))\\
&\quad - 2\varepsilon^{-2\mathfrak{y}-2}\widetilde{\varphi}^2(x)\mathcal{W}_{y_i}(\Theta_\varepsilon(X))\mathcal{W}_{y_n}(\Theta_\varepsilon(X))F_{x_i}(x')\\
&\quad + 2\varepsilon^{-2\mathfrak{y}-1}\widetilde{\varphi}_{x_i}(x)\widetilde{\varphi}(x)\mathcal{W}_{y_i}(\Theta_\varepsilon(X))\mathcal{W}(\Theta_\varepsilon(X))\\
&\quad - 2\varepsilon^{-2\mathfrak{y}-1}\widetilde{\varphi}_{x_i}(x)\widetilde{\varphi}(x)F_{x_i}(x')\mathcal{W}_{y_n}(\Theta_\varepsilon(X))\mathcal{W}(\Theta_\varepsilon(X))\\
&\quad + \varepsilon^{-2\mathfrak{y}-2}\widetilde{\varphi}^2(x)F^2_{x_i}(x')\mathcal{W}^2_{y_n}(\Theta_\varepsilon(X)) + \varepsilon^{-2\mathfrak{y}}\widetilde{\varphi}^2_{x_i}(x)\mathcal{W}^2(\Theta_\varepsilon(X))\Big]\\
&\quad + \varepsilon^{-2\mathfrak{y}-2}\widetilde{\varphi}^2(x)\mathcal{W}^2_{y_n}(\Theta_\varepsilon(X)) + 2\varepsilon^{-2\mathfrak{y}-1}\widetilde{\varphi}_{x_n}(x)\widetilde{\varphi}(x)\mathcal{W}_{y_n}(\Theta_\varepsilon(X))\mathcal{W}(\Theta_\varepsilon(X))\\
&\quad + \varepsilon^{-2\mathfrak{y}}\widetilde{\varphi}^2_{x_n}(x)\mathcal{W}^2(\Theta_\varepsilon(X)) + \varepsilon^{-2\mathfrak{y}-2}\widetilde{\varphi}^2(x)\mathcal{W}^2_z(\Theta_\varepsilon(X))\Bigg)\,dX\\
&=: J_1 - J_2 + \cdots + J_9 + J_{10}.
\end{aligned}$$

First, we estimate $J_1+J_7+J_{10}$ as follows:

$$\begin{aligned}
J_1+J_7+J_{10} &= \int_0^{+\infty} z^{1-2s}\int_{\mathbb{R}^n_+}\varphi_\delta^2(\varepsilon y)|\nabla_Y\mathcal{W}(Y)|^2\,dY\\
&= \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) - \int_{\mathbb{R}^n_+}\left[1-\varphi_\delta^2(\varepsilon y)\right]\cdot\mathcal{V}(y)\,dy.
\end{aligned}$$

From (39) we get

$$\int_{\mathbb{R}^n_+} \left[1-\varphi_\delta^2(\varepsilon y)\right] \cdot \mathcal{V}(y)\,dy \leqslant C\int_{\frac{\delta}{\varepsilon}}^{+\infty} r^{-3+2s-n}\,dr = C\left(\frac{\varepsilon}{\delta}\right)^{n-2s+2},$$

which gives

$$J_1 + J_7 + J_{10} = \mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+) + C(\delta)O(\varepsilon^{n-2s+2}).$$

Further, using (38) and (39) we estimate $J_3 + J_8$:

$$\begin{aligned} J_3 + J_8 &\leqslant 2\varepsilon\int_0^{+\infty} z^{1-2s}\int_{\mathbb{R}^n_+} \varphi_\delta(\varepsilon y)|\nabla_y\varphi_\delta(\varepsilon y)|\cdot\mathcal{W}(Y)|\nabla_Y\mathcal{W}(Y)|\,dY \\ &\leqslant \frac{C\varepsilon}{\delta}\left(\int_{\mathbb{K}_{\frac{\delta}{2\varepsilon}}}\mathcal{V}(y)\,dy\times\int_{\mathbb{K}_{\frac{\delta}{2\varepsilon}}}\int_0^{+\infty} z^{1-2s}|\mathcal{W}(Y)|^2\,dY\right)^{\frac12} \\ &\leqslant \frac{C\varepsilon}{\delta}\left(\int_{\frac{\delta}{2\varepsilon}}^{\frac{\delta}{\varepsilon}} r^{-3+2s-n}\,dr\times\int_{\frac{\delta}{2\varepsilon}}^{\frac{\delta}{\varepsilon}} r^{-1+2s-n}\,dr\right)^{\frac12} = C\left(\frac{\varepsilon}{\delta}\right)^{n-2s+2}. \end{aligned}$$

We estimate J_4 in a similar way:

$$\begin{aligned} |J_4| &\leqslant 2\varepsilon\int_0^{+\infty} z^{1-2s}\int_{\mathbb{R}^n_+}\varphi_\delta(\varepsilon y)|\nabla_y\varphi_\delta(\varepsilon y)|\mathcal{W}(Y)|\nabla_Y\mathcal{W}(Y)||\nabla_{y'}F(\varepsilon y')|\,dY \\ &\leqslant \frac{C\varepsilon}{\delta}\int_{\frac{\delta}{2\varepsilon}}^{\frac{\delta}{\varepsilon}} r^{2s-2n}\int_0^r \frac{\tau^{n-2}f_3(\varepsilon\tau)}{\sqrt{r^2-\tau^2}}d\tau dr \\ &\leqslant \frac{C\varepsilon^{n-2s+2}}{\delta}\int_{\frac{\delta}{2}}^{\delta}\tilde{r}^{2s-2n}\int_0^{\tilde{r}}\frac{\tilde{\tau}^{n-2}f_3(\tilde{\tau})}{\sqrt{\tilde{r}^2-\tilde{\tau}^2}}d\tilde{\tau}d\tilde{r} \\ &= C(\delta)\varepsilon^{n-2s+2}. \end{aligned}$$

Also, (38) allows us to estimate $J_6 + J_9$:

$$\begin{aligned} J_6 + J_9 &= C\varepsilon^2\int_0^{+\infty} z^{1-2s}\int_{\mathbb{R}^n_+}|\nabla_y\varphi_\delta(\varepsilon y)|^2\mathcal{W}^2(Y)\,dY \leqslant \frac{C\varepsilon^2}{\delta^2}\int_{\frac{\delta}{2\varepsilon}}^{\frac{\delta}{\varepsilon}} r^{-1+2s-n}\,dr \\ &= C\left(\frac{\varepsilon}{\delta}\right)^{n-2s+2}. \end{aligned}$$

Now we transform the main term J_2. Integrating by parts, we obtain

$$\begin{aligned} J_2 = -\frac{2}{\varepsilon}\int_0^{+\infty} z^{1-2s}\int_{\mathbb{R}^n_+}\sum_{i=1}^{n-1}\Big[&\varphi_\delta^2(\varepsilon y)\mathcal{W}_{y_iy_i}(Y)\mathcal{W}_{y_n}(Y)F(\varepsilon y') \\ &+\left[\varphi_\delta^2(\varepsilon y)\right]_{y_i}\mathcal{W}_{y_i}(Y)\mathcal{W}_{y_n}(Y)F(\varepsilon y') \\ &+\varphi_\delta^2(\varepsilon y)\mathcal{W}_{y_i}(Y)\mathcal{W}_{y_iy_n}(Y)F(\varepsilon y')\Big]\,dY. \end{aligned}$$

Next, we use the BVP (9) to express the sum of second derivatives:

$$J_2 = \frac{2}{\varepsilon}\int_0^{+\infty}\int_{\mathbb{R}^n_+}\varphi_\delta^2(\varepsilon y)\mathcal{W}_{y_n}(Y)\left[z^{1-2s}\mathcal{W}_{y_ny_n}(Y)+[z^{1-2s}\mathcal{W}_z(Y)]_z\right]F(\varepsilon y')\,dY$$
$$-\frac{2}{\varepsilon}\int_0^{+\infty}\int_{\mathbb{R}^n_+}z^{1-2s}\sum_{i=1}^{n-1}\left[\varphi_\delta^2(\varepsilon y)\right]_{y_i}\mathcal{W}_{y_i}(Y)\mathcal{W}_{y_n}(Y)F(\varepsilon y')\,dY$$
$$-\frac{1}{\varepsilon}\int_0^{+\infty}\int_{\mathbb{R}^n_+}z^{1-2s}\varphi_\delta^2(\varepsilon y)\left[|\nabla_{y'}\mathcal{W}(Y)|^2\right]_{y_n}F(\varepsilon y')\,dY =: \mathcal{H}+E_1+E_2.$$

Intergrating by parts once more, we transform $\mathcal{H}$ as follows:

$$\mathcal{H} = \frac{1}{\varepsilon}\int_0^{+\infty}\int_{\mathbb{R}^n_+}\varphi_\delta^2(\varepsilon y)F(\varepsilon y')\Big[z^{1-2s}\left[\mathcal{W}^2_{y_n}(Y)\right]_{y_n}$$
$$+2\mathcal{W}_{y_n}(Y)\left[z^{1-2s}\mathcal{W}_z(Y)\right]_z\Big]\,dY$$
$$= -\frac{1}{\varepsilon}\int_0^{+\infty}z^{1-2s}\Big[\int_{\mathbb{R}^{n-1}}\varphi_\delta^2(\varepsilon y')\mathcal{W}^2_{y_n}(y',0,z)F(\varepsilon y')dy'$$
$$+\int_{\mathbb{R}^n_+}[\varphi_\delta^2(\varepsilon y)]_{y_n}\mathcal{W}^2_{y_n}(Y)F(\varepsilon y')\,dy\Big]dz$$
$$+\frac{2\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)}{\varepsilon}\int_{\mathbb{R}^n_+}\varphi_\delta^2(\varepsilon y)F(\varepsilon y')\Phi_{y_n}(y)\frac{\Phi^{2^*_\sigma-1}(y)}{|y|^{(s-\sigma)2^*_\sigma}}\,dy$$
$$-\frac{1}{\varepsilon}\int_0^{+\infty}\int_{\mathbb{R}^n_+}\varphi_\delta^2(\varepsilon y)F(\varepsilon y')\left[z^{1-2s}\mathcal{W}_z^2(Y)\right]_{y_n}dY$$
$$=: -E_3+E_4+\mathcal{K}+E_7.$$

We integrate by parts $\mathcal{K}$ and E_7, taking into account $\mathcal{W}_z(y',0,z)=0$, and obtain

$$\mathcal{K} = \frac{2\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)}{\varepsilon}\int_{\mathbb{R}^n_+}\left[-\frac{\left[\varphi_\delta^2(\varepsilon y)\right]_{y_n}}{2^*_\sigma}+\varphi_\delta^2(\varepsilon y)\frac{(s-\sigma)y_n}{|y|^2}\right]\frac{\Phi^{2^*_\sigma}(y)}{|y|^{(s-\sigma)2^*_\sigma}}F(\varepsilon y')\,dy$$
$$=: E_5+E_6,$$
$$E_7 = \frac{1}{\varepsilon}\int_0^{+\infty}z^{1-2s}\int_{\mathbb{R}^n_+}\left[\varphi_\delta^2(\varepsilon y)\right]_{y_n}F(\varepsilon y')\mathcal{W}_z^2(Y)\,dY.$$

Lemma 8. *The following relations hold:*

(1) $|E_1+E_2+E_4+E_7| = C(\delta)\varepsilon^{n-2s+2}$;
(2)

$$\lim_{\varepsilon\to 0}\varepsilon\frac{E_3}{f(\varepsilon)} = C\int_0^{+\infty}\tau^{n+\alpha-2}\int_0^{+\infty}z^{1-2s}|\nabla_{\tau,z}\mathcal{W}(\tau,0,z)|^2\,dzd\tau < +\infty; \tag{67}$$

(3) $|E_5| = o(\varepsilon^{n-2s+2})$;
(4) $E_6 = \frac{2\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)}{2^*_\sigma}\mathcal{A}_1(\varepsilon)\cdot(1+o_\varepsilon(1))$.

Proof. (1) The statement follows from the following inequalities:

$$\begin{aligned}
|E_1+E_2+E_4+E_7| &= \bigg| \frac{1}{\varepsilon}\int_0^{+\infty} z^{1-2s}\int_0^{+\infty}\int_0^{+\infty}\Big[-2[\varphi_\delta^2(\varepsilon y)]_\tau \mathcal{W}_\tau(Y)\mathcal{W}_{y_n}(Y) \\
&\quad + [\varphi_\delta^2(\varepsilon y)]_{y_n}\left[\mathcal{W}_\tau^2(Y)-\mathcal{W}_{y_n}^2(Y)+\mathcal{W}_z^2(Y)\right]\Big]\,dy_n \\
&\quad \times \int_{\mathbb{S}_\tau^{n-2}} F(\varepsilon y')\,d\mathbb{S}_\tau^{n-2}(y')\,d\tau dz\bigg| \\
&\leqslant \frac{C}{\delta}\int_{\frac{\delta}{2\varepsilon}}^{\frac{\delta}{\varepsilon}}\int_0^r\int_0^{+\infty} z^{1-2s}\left|\nabla_{\tau,y_n,z}\mathcal{W}(\tau,\sqrt{r^2-\tau^2},z)\right|^2 dz\frac{r\tau^{n-2}}{\sqrt{r^2-\tau^2}}|f(\varepsilon\tau)|\,d\tau dr \\
&\leqslant \frac{C}{\delta}\int_{\frac{\delta}{2\varepsilon}}^{\frac{\delta}{\varepsilon}} r^{-2n+2s-1}\int_0^r \frac{\tau^{n-2}|f(\varepsilon\tau)|}{\sqrt{r^2-\tau^2}}\,d\tau dr \\
&= C(\delta)\varepsilon^{n-2s+2}\int_{\frac{\delta}{2}}^{\delta}\int_0^{\tilde r}\frac{\tilde r^{-2n+2s-1}\tilde\tau^{n-2}|f(\tilde\tau)|}{\sqrt{\tilde r^2-\tilde\tau^2}}\,d\tilde\tau d\tilde r.
\end{aligned}$$

(2) As for the estimate of I_3 in Lemma 7, we use the Lebesgue theorem: since

$$\begin{aligned}
\varepsilon\frac{E_3}{f(\varepsilon)} &= C\int_0^{+\infty}\tau^{n-2}\frac{f(\varepsilon\tau)}{f(\varepsilon)}\varphi_\delta^2(\varepsilon\tau)\int_0^{+\infty} z^{1-2s}|\nabla_{\tau,z}\mathcal{W}(\tau,0,z)|^2\,dzd\tau \\
&=: C\int_0^{+\infty} Q_\varepsilon(\tau)\,d\tau,
\end{aligned}$$

we get the integrand in the right-hand side of (67) as the pointwise limit of $Q_\varepsilon(\tau)$. To construct the majorant we use (39) and (66):

$$\begin{aligned}
Q_\varepsilon(\tau) &\leqslant \chi_{\left[0,\frac{\delta}{\varepsilon}\right]}(\tau)\tau^{n-2+\alpha}\frac{\psi(\varepsilon\tau)}{\psi(\varepsilon)}\int_0^{+\infty} z^{1-2s}|\nabla_{\tau,z}\mathcal{W}(\tau,0,z)|^2\,dz \\
&\leqslant \chi_{\left[0,\frac{\delta}{\varepsilon}\right]}(\tau)\frac{\psi(\varepsilon\tau)}{\psi(\varepsilon)}\frac{C\tau^{n-2+\alpha}}{1+|\tau|^{2n-2s+2}} \\
&\leqslant C(\delta)\left(\chi_{[0,1]}(\tau)\cdot\tau^{\alpha+n-\beta-2}+\chi_{[1,+\infty)}(\tau)\cdot\tau^{\alpha-n-4+\beta+2s}\right),
\end{aligned}$$

which is summable for sufficiently small β due to $\alpha < n-2s+3$.

(3) We have

$$\begin{aligned}
|E_5| &= \bigg|\frac{2\mathcal{S}_{s,\sigma}^{Sp}(\mathbb{R}_+^n)}{2_\sigma^*\cdot\varepsilon}\int_0^{+\infty}\int_{\mathbb{S}_\tau^{n-2}} F(\varepsilon y')\,d\mathbb{S}_\tau^{n-2}(y') \\
&\quad \times\int_0^{+\infty}\left[\varphi_\delta^2(\varepsilon\sqrt{\tau^2+y_n^2})\right]_{y_n}\frac{|\Phi|^{2_\sigma^*}(\tau,y_n)}{|y|^{(s-\sigma)2_\sigma^*}}\,dy_n d\tau\bigg| \\
&\leqslant \frac{C}{\delta}\int_{\frac{\delta}{2\varepsilon}}^{\frac{\delta}{\varepsilon}}\int_0^r\frac{|\Phi|^{2_\sigma^*}(\tau,\sqrt{r^2-\tau^2})}{r^{(s-\sigma)2_\sigma^*}}\frac{r\tau^{n-2}|f(\varepsilon\tau)|}{\sqrt{r^2-\tau^2}}\,d\tau dr \\
&\leqslant \frac{C}{\delta}\int_{\frac{\delta}{2\varepsilon}}^{\frac{\delta}{\varepsilon}}\int_0^r\frac{r\tau^{n-2}|f(\varepsilon\tau)|}{r^{2_\sigma^*(n-s-\sigma+1)}\sqrt{r^2-\tau^2}}\,d\tau dr \\
&\leqslant \frac{C\varepsilon^{2_\sigma^*(n-s-\sigma+1)-n}}{\delta}\int_{\frac{\delta}{2}}^{\delta}\tilde r^{1-2_\sigma^*(n-s-\sigma+1)}\int_0^{\tilde r}\frac{\tilde\tau^{n-2}|f(\tilde\tau)|}{\sqrt{\tilde r^2-\tilde\tau^2}}\,d\tau d\tilde r = o(\varepsilon^{n-2s+2}).
\end{aligned}$$

(4) Notice that the expression for E_6 coincides with the expression for I_3 up to two differences: we replace $\varphi_\delta^{2^*_\sigma}(\varepsilon y)$ with $\varphi_\delta^2(\varepsilon y)$ and multiply by $\frac{2\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)}{2^*_\sigma}$. Thus the statement follows from the argument from Lemma 7. $\square$

Lemma 8 together with estimates $I_3 \asymp f(\varepsilon)\varepsilon^{-1} \asymp E_3$ and $\varepsilon^{n-2s+2} = o(f(\varepsilon)\varepsilon^{-1})$ gives

$$J_2 = -E_3 \cdot (1 + o_\delta(1) + o_\varepsilon(1)) + \frac{2\mathcal{S}^{Sp}_{s,\sigma}(\mathbb{R}^n_+)}{2^*_\sigma}\mathcal{A}_1(\varepsilon) \cdot (1 + o_\varepsilon(1)).$$

It remains to estimate J_5. Using (39) and (43) we get

$$\begin{aligned} J_5 &= \int_0^{+\infty} z^{1-2s} \int_{\mathbb{R}^n_+} \varphi_\delta^2(\varepsilon y)|\nabla_{y'} F(\varepsilon y')|^2 \mathcal{W}^2_{y_n}(Y)\, dY \\ &\leqslant C \int_0^{\frac{\delta}{\varepsilon}} \tau^{n-2} f_2(\varepsilon\tau)\, d\tau \int_0^{\sqrt{\frac{\delta^2}{\varepsilon^2}-\tau^2}} \mathcal{V}(\tau, y_n)\, dy_n \\ &\leqslant \int_0^{\frac{\delta}{\varepsilon}} \int_0^{+\infty} \frac{C\tau^{n-2} f_2(\varepsilon\tau)}{(1+\tau^2+y_n^2)^{n-s+1}}\, dy_n d\tau \\ &\leqslant \int_0^{\frac{\delta}{\varepsilon}} \frac{C\tau^{n-2} f_2(\varepsilon\tau)}{(1+\tau^2)^{\frac{2n-2s+1}{2}}}\, d\tau = \frac{o_\delta(1)}{\varepsilon} \int_0^{\frac{\delta}{\varepsilon}} \frac{\tau^{n-3}|f(\varepsilon\tau)|}{(1+\tau^2)^{\frac{2n-2s+1}{2}}}\, d\tau. \end{aligned}$$

The last integral can be estimated in the same way as E_3 in Lemma 8. This estimate gives $J_5 = o_\delta(1)E_3$.

Denoting $\mathcal{A}_2(\varepsilon) := E_3$, we obtain (52).

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