A Note on Random Greedy Coloring of Uniform Hypergraphs

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Received 3 October 2013; accepted 21 January 2014
Published online 30 May 2014 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/rsa.20556

ABSTRACT: The smallest number of edges forming an n-uniform hypergraph which is not r-colorable is denoted by $m(n, r)$. Erdős and Lovász conjectured that $m(n, 2) = \Theta(n2^n)$. The best known lower bound $m(n, 2) = \Omega\left(\sqrt{n/\ln(n)}2^n\right)$ was obtained by Radhakrishnan and Srinivasan in 2000. We present a simple proof of their result. The proof is based on the analysis of a random greedy coloring algorithm investigated by Pluhár in 2009. The proof method extends to the case of r-coloring, and we show that for any fixed r we have $m(n, r) = \Omega\left((n/\ln(n))^{(r-1)/r}\ r^n\right)$ improving the bound of Kostochka from 2004. We also derive analogous bounds on minimum edge degree of an n-uniform hypergraph that is not r-colorable. © 2014 Wiley Periodicals, Inc. Random Struct. Alg., 47, 407–413, 2015

Keywords: coloring; hypergraph; greedy algorithm

1. INTRODUCTION

A hypergraph is a pair (V, E), where V is a set of vertices and E is a family of subsets of V. Hypergraph is *n-uniform* if all its edges have exactly n elements. Hypergraph (V, E) is *r-colorable* if there exists a coloring of vertices with r colors in which no edge is monochromatic (i.e., there exists a function $c : V \to \{1, \dots, r\}$ such that the image of every edge has at least two elements). Hypergraph has *property B* if it is two-colorable. For $n, r \in \mathbb{N}$ let $m(n, r)$ be the smallest number of edges of an n-uniform hypergraph that is not r-colorable. The asymptotic behaviour of $m(n) = m(n, 2)$ was first studied by Erdős. In [1] and [2]

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*Supported by Polish National Science Center (grant 2011/01/D/ST1/04412).
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Erdős proved that:

$$2^{n-1} \leq m(n) \leq (1 + o(1))\frac{e \ln 2}{4} n^2 2^n.$$

In [3] Erdős and Lovász wrote that "perhaps $n2^n$ is the correct order of magnitude of $m(n)$". The upper bound has not been improved since. The most recent improvement on the lower bound was obtained by Radhakrishnan and Srinivasan in [7]. We present a simple proof of their main theorem:

Theorem 1 ([7])**.**

$$m(n) = \Omega\left(\left(\frac{n}{\ln(n)}\right)^{1/2} 2^n\right).$$

In fact we prove that, for $c < \sqrt{2}$ and all sufficiently large n, whenever an n-uniform hypergraph has at most $c\sqrt{n/\ln(n)}2^{n-1}$ edges, then a simple random greedy algorithm produces a proper coloring with positive probability. The same coloring procedure was considered by Pluhár in [5]. In an elegant and straightforward way he proved that for some specific constant $c > 0$ a bound $m(n) > cn^{1/4}2^n$ is valid for all $n > 1$.

The proof technique extends easily to the more general case of r-coloring (very much along the lines of development of Pluhár [5]). To avoid technicalities we focus on asymptotics of $m(n, r)$ for a fixed r and n tending to infinity.

Theorem 2. *For any fixed integer $r \geq 2$, we have*

$$m(n, r) = \Omega\left(\left(\frac{n}{\ln(n)}\right)^{\frac{r-1}{r}} r^n\right).$$

This improves the bounds of Kostochka [4] which are of the order $(\frac{n}{\ln(n)})^{\frac{\lfloor \log_2(r) \rfloor}{\lfloor \log_2(r) \rfloor + 1}} r^n$ and the bound $m(n, 3) = \Omega(n^{1/2}3^{n-1})$ by Shabanov [9]. Several other variants of extremal problems on hypergraph coloring can be found in a survey by Raigorodskii and Shabanov [8].

Just like the results from [7] our results extend to a local version. Let $D(n, r)$ be the maximum number such that every n-uniform hypergraph with strictly smaller edge degrees is r-colorable.

Theorem 3. *For any fixed $r \geq 2$ we have*

$$D(n, r) = \Omega\left(\left(\frac{n}{\ln(n)}\right)^{\frac{r-1}{r}} r^n\right).$$

All results are derived from the analysis of a random greedy r-coloring procedure presented as Algorithm 1.

Random value $t(v)$ assigned by the algorithm to a vertex v will be called a *birth time* of v. We assume that the birth time assignment function sampled by the algorithm is injective (this happens with probability 1). For any edge f, the *first* (resp. *last*) vertex of f is the vertex $v \in f$ with smallest (largest) birth time.

Algorithm 1: Random greedy r-coloring

foreach $v \in V$ **do**
  choose uniformly and independently at random a point $t(v)$ from the interval $[0, 1]$
let $(v_1, \ldots, v_m)$ **be** V ordered according to $t(v)$ (i.e. $t(v_i) \leq t(v_{i+1})$)
for $i = 1 \ldots m$ **do**
  if $\exists_{j \in \{1,\ldots,r\}}$ *such that coloring v_i with j does not create a monochromatic edge with its highest-index vertex being v_i* **then**
    $c(v_i) \leftarrow$ smallest such j
  else
    $c(v_i) \leftarrow r$
return c

2. PROPERTY B

Proof of Theorem 1. Let (V, E) be an n-uniform hypergraph with $k2^{n-1}$ edges. Let us consider Algorithm 1 with $r = 2$. Following a long tradition, we call colors $1, 2$ respectively *blue* and *red*. Then the rule of assigning colors used by the algorithm reduces to *"choose color blue unless the currently colored vertex is the last vertex of a blue edge."* Every pair of edges (e,f) such that the last vertex of e is the first vertex of f will be called a *conflicting pair*.

Clearly there are no monochromatic blue edges in the coloring constructed by the algorithm. Suppose that some edge f is colored red, and let v be the first vertex of f. Vertex v has been colored red by the algorithm, so there exists an edge e, such that v is the last vertex of e. Edges (e,f) form a conflicting pair. By the above discussion if there are no conflicting pairs under the assignment t, then the coloring produced by the algorithm is proper. We are going to check for which values of k the probability of having no conflicting pairs is positive.

Let us divide real interval $[0, 1]$ into three subintervals $B = \left[0, \frac{1-p}{2}\right), P = \left[\frac{1-p}{2}, \frac{1+p}{2}\right), R = \left[\frac{1+p}{2}, 1\right]$ (with parameter p to be optimized later). We consider three events:

B: there exists a conflicting pair with a common vertex in B,
P: there exists a conflicting pair with a common vertex in P,
R: there exists a conflicting pair with a common vertex in R.

Clearly $\Pr[\mathbf{B}] = \Pr[\mathbf{R}]$ and they are both smaller than the probability that there exists an edge such that all its vertices have birth times from interval B. The expected number of such edges is $k2^{n-1}(\frac{1-p}{2})^n$, hence:

$$\Pr[\mathbf{B} \cup \mathbf{R}] \leq \Pr[\mathbf{B}] + \Pr[\mathbf{R}] = 2\Pr[\mathbf{B}] \leq k2^n \left(\frac{1-p}{2}\right)^n = k(1-p)^n. \tag{2.1}$$

A pair of edges (e,f) with exactly one common vertex is called *dangerous*. Only dangerous pairs can be conflicting. The probability that there exists a conflicting pair with common vertex in P is bounded from above by the expected number of such pairs:

$$\begin{aligned}\Pr[\mathbf{P}] &\le \mathbb{E}[\sharp \text{ conflicting pairs with a common vertex in P}] \\ &\le (k2^{n-1})^2 \Pr[\text{dangerous pair (e,f) is conflicting with a common vertex in P}] \\ &\le (k2^{n-1})^2 \int_{\frac{1-p}{2}}^{\frac{1+p}{2}} x^{n-1}(1-x)^{n-1}\mathrm{d}x = k^2 \int_{-\frac{p}{2}}^{\frac{p}{2}} ((1+2x)(1-2x))^{n-1}\,\mathrm{d}x\end{aligned}$$

The integrand function is smaller than 1, so the length of the integration interval is an upper bound for the value of the integral. We get

$$\Pr[\mathbf{P}] \le k^2 p. \tag{2.2}$$

Inequalities (2.1) and (2.2) give:

$$\Pr[\mathbf{B} \cup \mathbf{R} \cup \mathbf{P}] \le \Pr[\mathbf{B} \cup \mathbf{R}] + \Pr[\mathbf{P}] \le k(1-p)^n + k^2 p.$$

Hence, whenever the following inequality holds

$$k(1-p)^n + k^2 p < 1, \tag{2.3}$$

the algorithm produces a proper coloring with positive probability.

Let $k_n = c\sqrt{n/\ln(n)}$ and $p_n = \ln(n/k_n)/n$. Then

$$\lim_{n\to\infty} \left(k_n(1-p_n)^n + k_n^2 p_n\right) = c^2/2.$$

Therefore for any $c < \sqrt{2}$ and all sufficiently large n, any n-uniform hypergraph with at most $c\sqrt{n/\ln(n)}2^{n-1}$ edges has property B. ∎

3. *r*- COLORING

Proof of Theorem 2. Let (V, E) be an n-uniform hypergraph with $k\ r^{n-2}$ edges. We analyse the probability that Algorithm 1 working with r colors produces a proper coloring of the hypergraph. Analogously to the developments of Pluhár [5] we focus on avoiding specific structures called conflicting r-chains.

A sequence of edges $(f_1, \dots, f_r)$ is called an *r-chain* if $|f_i \cap f_{i+1}| = 1$ for each $i \in \{1, \dots, r-1\}$, and $f_i \cap f_j = \emptyset$ for all $i, j \in \{1, \dots r\}$ such that $|i-j| > 1$. An r-chain is *conflicting* under birth time assignment t if for each $i \in \{1, \dots, r-1\}$ the last vertex of f_i is the first vertex of f_{i+1}. It is easy to check that all monochromatic edges in the coloring constructed by the algorithm have color r and every such edge is the last edge of some conflicting r-chain. Therefore, if there are no conflicting r-chains, then the coloring produced by the algorithm is proper.

Let us set $p = \frac{2\ln(n)}{n}$. The *length* of an edge $f \in E$ (under the assignment t) is the minimum length of an interval containing all the birth times of the vertices of f. An edge is *short* if its length is smaller than $\frac{1-p}{r}$, otherwise the edge is *long*. The expected number of short edges is smaller than

$$k\ r^{n-2}\ n \left(\frac{1-p}{r}\right)^{n-1} \sim \frac{k}{r\ n}. \tag{3.1}$$

Next we estimate the probability of observing a conflicting r-chain in which no edge is short. Let $F = (f_1, \dots, f_r)$ be an r-chain, and $x_1, \dots, x_{r-1}$ be vertices such that $f_i \cap f_{i+1} = \{x_i\}$. Observe that for F to be conflicting without short edges, the birth time of each x_i must belong to the interval $[\frac{i-ip}{r}, \frac{i+(r-i)p}{r}]$ (otherwise the average length of edges to the left or to the right would be smaller than $\frac{1-p}{r}$). The probability that vertices $x_1, \dots, x_{r-1}$ have birth times in corresponding intervals is p^{r-1}. Once those birth times are fixed, the probability that remaining vertices of the chain fall into appropriate intervals is smaller than (for convenience we put $t(x_0) = 0, t(x_r) = 1$)

$$\prod_{i=0}^{r-1} (t(x_{i+1}) - t(x_i))^{n-2}.$$

Since the sum of differences in the product is 1, the product is maximized when $t(x_{i+1}) - t(x_i) = 1/r$ for all $i \in \{0, \dots, r-1\}$. Hence the probability that an r-chain is conflicting is less than $p^{r-1} r^{-r(n-2)}$. As a consequence the expected number of conflicting r-chains without short edges is less than

$$\frac{2}{r!}(k\ r^{n-2})^r p^{r-1} r^{-r(n-2)} = \frac{2}{r!} k^r \left(\frac{2\ln(n)}{n} \right)^{r-1}. \tag{3.2}$$

For $k < (\frac{n}{2\ln(n)})^{\frac{r-1}{r}}$, that number is smaller than $\frac{2}{r!}$. Moreover, if n is large enough, then the expected number of short edges (3.1) is close to zero. In those cases the algorithm produces a proper r-coloring with positive probability and the theorem follows. ∎

Corollary 4. *If there exists a birth time assignment which makes no short edge and creates no conflicting r-chain of long edges, then Algorithm 1, working with r colors, produces a proper coloring with positive probability (at least the probability of sampling such birth time assignment).*

4. LOCAL VERSION

Proof of Theorem 3. Let $H = (V, E)$ be an n-uniform hypergraph with maximum edge degree $D - 1$. To derive sufficient condition for H to be r-colorable we apply Lovász Local Lemma to prove that there exists a birth time assignment avoiding short edges and conflicting r-chains of long edges. Then, by Corollary 4, Algorithm 1 working with r colors produces a proper coloring with positive probability. For a birth assignment function t chosen uniformly at random (as in Algorithm 1) let us consider the following events:

(1) let $\mathcal{S}_f$ be the event that edge f is short,
(2) let $\mathcal{C}_s$ be the event that an r-chain s consists of only long edges and is conflicting.

The values of $P_1 = \Pr(\mathcal{S}_f)$ and $P_2 = \Pr(\mathcal{C}_f)$ were analysed in Section 3. Clearly every event $\mathcal{S}_f$ is independent of all events $\mathcal{S}_e$ and $\mathcal{C}_s$ for e and s disjoint from f (analogously for events $\mathcal{C}_s$). Every edge intersects at most D edges and rD^r different r-chains. Similarly an r-chain intersects at most rD edges and r^2D^r r-chains. Therefore it is sufficient to exhibit $x, y \in [0, 1)$ for which:

$$P_1 \leq x(1-x)^D (1-y)^{rD^r} \quad \text{and} \quad P_2 \leq y(1-x)^{rD} (1-y)^{r^2D^r}$$

to conclude from Lovász Local Lemma that there exists a birth time assignment function which avoids short edges and conflicting r-chains of long edges. Choosing $x = 1 - e^{-a/D}$ and $y = 1 - e^{-b/(rD^r)}$ the right hand sides of the inequalities become $xe^{-(a+b)}$ and $ye^{-r(a+b)}$. A tedious and standard calculation, that we omit here, shows that it is possible to choose positive a, b, c so that the inequalities are satisfied for all large enough n and $D < c(\frac{n}{\ln(n)})^{(r-1)/r} r^n$. ∎

5. REMARKS

(1) Inequality (2.3) is exactly the inequality optimized in [7].
(2) The optimal value for p in the case of 2-coloring has the following interpretation. Suppose that the birth time of the last vertex of an edge e is $\frac{1-p}{2}$. Then conditional expected number of conflicting pairs (e,f) is at most $k2^{n-1}n^{-1}(\frac{1+p}{2})^{n-1}$, which tends to 1 with n, for chosen k and p.
(3) The birth times of vertices are used in Algorithm 1 only to generate an ordering of V. Therefore the same results apply to an algorithm which instead chooses uniformly at random a permutation of V.
(4) Careful analysis of the algorithm which chooses random permutation can give essentially better bound when the number of vertices is sufficiently small. In particular, considered algorithms are never worse than choosing equitable an partition of vertices into color classes. As observed in [6] for $|V| = O(n^2/\ln(n))$ the last strategy with positive probability constructs a proper 2-coloring of hypergraphs with at most $\Theta(n2^n)$ edges.
(5) The presented analysis of Algorithm 1 shows that within some intervals the ordering of vertices is irrelevant (e.g. in intervals B, R for 2-coloring). It suggests an equivalent variant of the algorithm which first chooses vertices which fall into these intervals, color these vertices accordingly, and then use random greedy coloring for the remaining ones. Those two phases can be considered as precoloring and random alteration. For 2-coloring it closely resembles the algorithm of Radhakrishnan and Srinivasan from [7] (especially the simplification by Boppana mentioned in the paper).
(6) Random greedy coloring algorithm easily translates to a streaming framework analysed in [6].

ACKNOWLEDGMENTS

The present paper is a result of combining two independent papers by the authors each one containing Theorem 2 as the main result. The authors are grateful to N. Alon, A.M. Raigorodskii, A. Ruciński, D.A. Shabanov, and J. Spencer for discussions and valuable suggestions.

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