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REGULAR BEHAVIOR OF THE MAXIMAL HYPERGRAPH CHROMATIC NUMBER*

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Abstract. Let $m(n, r)$ denote the minimal number of edges in an n-uniform hypergraph which is not r-colorable. It is known that for a fixed n one has $c_n r^n < m(n, r) < C_n r^n$. We prove that for any fixed n the sequence $a_r := m(n, r)/r^n$ has a limit, which was conjectured by Alon. We also prove the list colorings analogue of this statement.

Key words. hypergraph coloring, subadditivity, Erdős–Hajnal problem

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1. Introduction.

A hypergraph $H = (V, E)$ consists of a finite set of *vertices* V and a family E of the subsets of V, which are called *edges*. A hypergraph is called n-*uniform* if every edge has size n. A *vertex* r-*coloring* of a hypergraph $H = (V, E)$ is a map from V to $\{1, \ldots, r\}$. A coloring is *proper* if there is no monochromatic edge, i.e., any edge $e \in E$ contains two vertices of different color. The *chromatic number* of a hypergraph H is the smallest number $\chi(H)$ such that there exists a proper $\chi(H)$-coloring of H. Let $m(n, r)$ be the minimal number of edges in an n-uniform hypergraph with chromatic number more than r.

We are interested in the case when n is much smaller than r (see [9, 8] for the general case and related problems).

1.1. Upper bounds.

For $n = 2$ (i.e., for graphs) the problem of finding $m(n, r)$ is trivial. Indeed, $m(2, r) \geq \binom{r+1}{2}$ since any coloring of a given G in $\chi(G)$ colors should contain an edge between every pair of colors, otherwise one can join these two colors, so G can be properly colored by $\chi(G)-1$ colors, which is absurd. On the other hand, the complete graph on $r + 1$ vertices gives an example.

Erdős conjectured [4] that

$$m(n, r) = \binom{(n-1)r+1}{n}$$

for $r > r_0(n)$, that is achieved by the complete hypergraph on $(n - 1)r + 1$ vertices.

However Alon [2] disproved the conjecture for n large enough by using the estimate

$$m(n, r) \leqslant \min_{a \geq 0} T(r(n + a - 1) + 1, n + a, n),$$

where the Turán number $T(v, k, n)$ is the smallest number of edges in an n-uniform hypergraph on v vertices such that every induced subgraph on k vertices contains an

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edge. Different bounds on Turán numbers beat the complete n-uniform hypergraph construction when $n > 3$ (see [10] for a survey). So the case $n = 3$ is in some sense the most interesting.

Using the same inequality with better bounds on Turán numbers, Akolzin and Shabanov [1] showed that

$$m(n, r) < Cn^3 \ln n \cdot r^n.$$

Alon [2] conjectured that for a fixed n the quantity $m(n, r)$ has regular behavior, i.e., the sequence $m(n, r)/r^n$ has a limit.

1.2. Lower bounds. There are several ways to show an inequality of type $m(n, r) > c(n)r^n$. Alon [2] uses an alteration-type trick to get the first bound of such type:

$$m(n, r) \geq (n-1) \left\lceil \frac{r}{n} \right\rceil \left\lfloor \frac{n-1}{n} r \right\rfloor^{n-1}.$$

Pluhár's random greedy approach [7] gives the bound

$$m(n, r) > c\sqrt{n}r^n$$

as noted in [9]. Finally, combining two previous arguments Akolzin and Shabanov [1] proved that

$$m(n, r) > c\frac{n}{\ln n} r^n.$$

1.3. List colorings. Let $H = (V, E)$ be a hypergraph and let $\{L(v)\}$, $v \in V(H)$, be sets; we refer to these sets as *lists*. A *list coloring* of H is an assignment of a color from $L(v)$ to each $v \in V(H)$; a list coloring is *proper* if there is no monochromatic edge. The *list chromatic number* of a hypergraph H is the minimal k such that for any assignment of lists $L(v)$, each of size k, there exists a proper list coloring. Define the quantity $m_c(n, r)$ as the minimal number of edges of an n-uniform hypergraph with list chromatic number greater than r.

By definition, $m_c(n, r) \leq m(n, r)$, and this is the only known upper bound on $m_c(n, r)$ (also, it is not known whether $m_c(n, r) = m(n, r)$ for all n, r).

It was recently proved by B. Sudakov (unpublished) that there is $c > 0$ such that

$$m_c(n, r) \geq cr^n$$

for all n, $r > r_0(n)$.

Structure of the paper. Section 2 contains the proof of the Alon conjecture that the sequence $a_r := m(n, r)/r^n$ has a limit. Section 3 proves the same result for $m_c(n, r)$. The final section consists of open questions.

2. Colorings. Fix $n > 1$ and denote by $f(N)$ the maximal possible chromatic number of an n-uniform hypergraph with N edges, in particular, $f(0) = 1$. The function $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}_{\geqslant 1}$ nonstrictly increases and satisfies

$$m(n, r) = \min\{N : f(N) > r\}.$$

Therefore $m(n, r) \sim Cr^n$ if and only if $f(N) \sim (N/C)^{1/n}$.

Here is the crucial lemma.

LEMMA 1. *For any $N > 0$ and any positive integer p we have*

$$f(N) \leqslant \max_{a_1+a_2+\cdots+a_p \leqslant N/p^{n-1}} f(a_1) + f(a_2) + \cdots + f(a_p). \tag{2.1}$$

Proof. Let $H = (V, E)$ be an n-uniform hypergraph with $|E| = N$.

Choose the auxiliary colors $\eta(v) \in \{1, 2, \dots, p\}$ at random uniformly and independently and denote $V_i = \eta^{-1}(\{i\})$. Let $H_i = (V_i, E_i)$ be the hypergraph induced by H on V_i. The expectation of $\sum_{i=1}^{p} |E_i|$ equals $|E|/p^{n-1}$ because each edge of H belongs to some H_i with the same probability $1/p^{n-1}$. Therefore there exists a certain auxiliary coloring η such that

$$\sum |E_i| \leqslant N/p^{n-1}.$$

Fix such a coloring η and properly color each H_i using $f(|E_i|)$ colors, using disjoint sets of colors for different i. In total we use $\sum f(|E_i|)$ colors and H is colored properly.

Since H is an arbitrary n-uniform hypergraph with N edges the proof is completed. ∎

The rest of the proof is completely analytical; all combinatorics are in Lemma 1. Namely, the following general statement holds.

THEOREM 1. *Assume that $n > 1$ is a fixed integer, $N_0 > 0$ is a constant, $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}_{>0}$ is a function satisfying (2.1) for all $N \geqslant N_0$, and $p \in \{2, 3\}$. Then*

$$\lim_{x\to\infty} \frac{f(x)}{x^{1/n}}$$

exists and is finite.

To prove Theorem 1 we use the following lemma.

LEMMA 2. *Denote $c_n = \lceil (1 - 2^{1/n-1})^{-n} \rceil$. Under the conditions of Theorem 1 for any $M \geqslant N_0$ the inequality*

$$f(N) \leqslant N^{1/n} \cdot \max_{M \leqslant a < c_n M} f(a) \cdot a^{-1/n}$$

holds for all $N \geqslant M$.

Proof. Do an induction on $N \in \{M, M+1, \dots\}$. The base $N < c_n M$ is clear.

The induction steps from $M, M+1, \dots, N-1$ to N assuming $N \geqslant c_n M$.

Denote

$$\lambda = \max_{M \leqslant a < c_n M} f(a) \cdot a^{-1/n}.$$

By (2.1) with $p = 2$ we have $f(N) \leqslant f(a) + f(b)$ for certain nonnegative integers a, b such that $a + b \leqslant N/2^{n-1}$. If $\min(a, b) \geqslant M$, then by the induction proposition we get

$$f(a) + f(b) \leqslant \lambda(a^{1/n} + b^{1/n}) \leqslant 2\lambda \left(\frac{a+b}{2}\right)^{1/n} \leqslant \lambda N^{1/n},$$

as desired. If, for example, $a < M$, we get

$$f(a) + f(b) \leqslant f(M) + f(b) \leqslant \lambda \left(M^{1/n} + \left(\frac{N}{2^{n-1}}\right)^{1/n} \right) \leqslant \lambda N^{1/n}$$

provided that $N \geqslant c_n M$. ∎

Lemma 2 in particular implies that the maxima $M(k)$ of the function $g(x) := f(x)x^{-1/n}$ over the segments $[c_n^k, c_n^{k+1}]$ eventually (for $k \geqslant k_0$) do not increase. Let α_0 denote the limit of $M(k)$; it is also the upper limit of the function g.

Fix p in Lemma 1.

Further we need the following standard technical proposition.

Proposition 1. *For any $\theta > 1$ there exists $\delta > 0$ such that for all nonnegative real numbers $x_1, \dots, x_p$ with the arithmetic mean $x_0 = (x_1 + \dots + x_p)/p$ the inequality*

$$\sum_{i=1}^{p} x_i^{1/n} \geqslant (p - \delta) \cdot x_0^{1/n}$$

yields $x_i \in [x_0/\theta, x_0 \cdot \theta]$.

Proof. The case $x_0 = 0$ is clear. If $x_0 > 0$, denote $y_i = x_i/x_0$; then $\sum y_i = p$ and $\sum y_i^{1/n} \geqslant p - \delta$. Let $\ell(x) = 1 + (x-1)/n$ be a tangent line to the graph of the function $x^{1/n}$ at point $(1, 1)$. We have $\sum \ell(y_i) = p$. By concavity we have $y^{1/n} \leqslant \ell(y)$ with equality only at $y = 1$, and for given $\theta > 1$ there exists $\delta > 0$ such that $\ell(y) - y^{1/n} > \delta$ for $y \notin [1/\theta, \theta]$. Therefore

$$\delta \geqslant p - \sum_{i=1}^{p} y_i^{1/n} = \sum_{i=1}^{p} \left(\ell(y_i) - y_i^{1/p} \right),$$

all summands $\ell(y_i) - y_i^{1/p}$ belong to $[0, \delta]$, and therefore $y_i \in [1/\theta, \theta]$ and $x_i \in [x_0/\theta, x_0\theta]$. ∎

We proceed with the proof of Theorem 1.

Let N be a large integer with $g(N) = \alpha_0 + o(1)$. In other words, N grows to infinity along such a subsequence that $g(N)$ converges to α_0. Find for this N the numbers $a_1, \dots, a_p$ as in Lemma 1. Note that for any $\varepsilon > 0$ there exists $C > 0$ such that $f(a) \leqslant (\alpha_0 + \varepsilon)a^{1/n} + C$ for all integers $a \geqslant 0$. It follows that $f(a) \leqslant \alpha_0 a^{1/n} + o(N^{1/n})$ uniformly for all $a \leqslant N$. Therefore

$$\alpha_0 \cdot p \cdot \left(\frac{a_1 + \dots + a_p}{p} \right)^{1/n} \leqslant \alpha_0 N^{1/n} = f(N) + o(N^{1/n}) \leqslant \alpha_0 \sum_{i=1}^{p} a_i^{1/n} + o(N^{1/n}).$$

So all inequalities here are equalities with accuracy $o(N^{1/n})$. In particular $\sum a_i = N/p^{n-1} + o(N)$ and all a_i are asymptotically equal to $N/p^n + o(N)$ by Proposition 1. Also $f(a_i) = \alpha_0 N^{1/n}/p + o(N^{1/n})$ for all $i = 1, \dots, p$. Equivalently, $g(a_i) = \alpha_0 + o(1)$ for all $i = 1, \dots, p$.

Consider the numbers of the form $2^{nx}3^{ny}$ with nonnegative integer x, y; call them *appropriate* numbers.

So we proved that for large N with $g(N) = \alpha_0 + o(1)$ there exists $\tilde{N} = N/p^n + o(N)$ with $g(\tilde{N}) = \alpha_0 + o(1)$. Consecutively using this for $p \in \{2, 3\}$ we conclude that whenever $g(N) = \alpha_0 + o(1)$ and R is appropriate, then there exists $a = N/R + o(N)$ such that $g(a) = \alpha_0 + o(1)$.

The ratio of two consecutive appropriate numbers tends to 1 by the basic Dirichlet–Kronecker Diophantine approximation lemma. Fix $\rho > 1$ and choose appropriate numbers $r_1 < r_2 < \dots < r_m$ so that $r_{i+1}/r_i < \rho$, but $r_1 < c_n^S$, $r_m > c_n^{S+10}$ for certain positive integer S.

So we may find numbers $N_i = N/r_i + o(N)$ such that $g(N_i) = \alpha_0 + o(1)$ for all $i = 1, 2, \dots, m$.

For large k choose $N \in [c_n^k, c_n^{k+1}]$ with maximal possible value $g(N)$; we have $g(N) = \alpha_0 + o(1)$. For any integer number x in the segment $[c_n^{k-S-2}, c_n^{k-S-1}]$ choose minimal i such that $x > N_i$. Then $x \leqslant N_i \cdot \rho$ and

$$f(x) \geqslant f(N_i) = (\alpha_0 + o(1))N_i^{1/n} \geqslant (\alpha_0 + o(1))(x/\rho)^{1/n}.$$

Therefore

$$\liminf f(x)x^{-1/n} \geqslant \alpha_0 \rho^{-1/n},$$

and since $\rho > 1$ was arbitrary, the lower limit of the function $g(x) = f(x)x^{-1/n}$ equals its upper limit α_0. This completes the proof of Theorem 1.

Theorem 1 and Lemma 1 immediately yield the following.

THEOREM 2. *For fixed n, the sequence $m(n, r)/r^n$ has a limit.*

3. List colorings.

Here we prove the choice version of Theorem 2.

THEOREM 3. *For fixed integer $n > 1$ the sequence $m_c(n, r)/r^n$ has a finite positive limit.*

Denote by $f_c(N)$ the maximal possible list chromatic number of an n-uniform hypergraph with N edges. Since the list chromatic number is always not less than the chromatic number, we get

$$f_c(N) \geqslant \delta N^{1/n} \tag{3.1}$$

for certain $\delta > 0$ depending only on n. Theorem 3 is equivalent to the existence of a finite limit of $f_c(N)/N^{1/n}$.

We use the following Chernoff-type concentration inequality for the sum of independent $\{0, 1\}$-valued random variables.

PROPOSITION 2. *If n is a positive integer and $\xi_1, \ldots, \xi_n$ are independent random variables taking values in $\{0, 1\}$, A is the expectation of $S := \sum_{i=1}^n \xi_i$, $T \in [0, A]$, then*

$$\text{prob}\{S \leqslant A - T\} \leqslant e^{-\frac{T^2}{2A}}.$$

See the proof, for example, in [6, Theorem 4.5].

We need the following technical statements.

LEMMA 3. *Assume that $n > 1$ is a fixed integer; $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}_{>0}$ is a function satisfying*

$$f(x) \leqslant \max_{a+b \leqslant x/2^{n-1}} f(a) + f(b) + M(f(a)^\alpha + f(b)^\alpha) \quad \forall x \geqslant x_0 \tag{3.2}$$

for certain constants $x_0 > 0$, $\alpha \in (0, 1)$, $M > 0$. Then $f(x) = O(x^{1/n})$ for large x.

Proof. We recursively define the increasing sequence $h_0 \leqslant h_1 \leqslant \cdots$ of positive numbers such that

$$f(x) \leqslant h_k \cdot x^{1/n} \quad \text{for} \quad 1 \leqslant x \leqslant x_0 \cdot 2^{(n-1)k}. \tag{3.3}$$

Choose h_0 large enough (so that $h_0 > 1$, (3.3) for $k = 0$ is satisfied, and also something else, to be specified later, holds). Assume that $k \geqslant 1$ and (3.3) holds for $0, 1, \ldots, k-1$. Choose $x \in (x_0 \cdot 2^{(n-1)(k-1)}, x_0 \cdot 2^{(n-1)k}]$. This x satisfies (3.2). Fix corresponding a, b and consider two cases: either $\min(a, b) = 0$ or both a, b are positive.

In the first case we get

$$f(x) \leqslant h_{k-1}(2^{1-n}x)^{1/n} + Mh_{k-1}^{\alpha}(2^{1-n}x)^{\alpha/n} + f(0) + M(f(0))^{\alpha}. \tag{3.4}$$

If h_{k-1} is large enough, the right-hand side does not exceed $h_{k-1}x^{1/n}$. This may be guaranteed by choosing large enough h_0.

In the second case both a and b satisfy the induction hyphothesis and we get

$$f(x) \leqslant h_{k-1}(a^{1/n} + b^{1/n}) + 2Mh_{k-1}^{\alpha}(2^{1-n}x)^{\alpha/n} \leqslant h_{k-1}x^{1/n} + 2Mh_{k-1}^{\alpha}(2^{1-n}x)^{\alpha/n}. \tag{3.5}$$

The right-hand side of (3.5) does not exceed

$$h_{k-1}x^{1/n}\left(1 + 2Mx^{(\alpha-1)/n}\right).$$

Since $x \geqslant x_0 \cdot 2^{(n-1)(k-1)}$, it allows us to choose

$$h_k = h_{k-1}\left(1 + 2Mx_0^{(\alpha-1)/n} \cdot 2^{(\alpha-1)(k-1)(n-1)/n}\right)$$

and (3.3) for k holds. The sequence h_k obviously increases and the sequence $h_k/h_{k-1} - 1$ decays exponentially. Thus the infinite product of h_k/h_{k-1} converges, i.e., h_k is bounded. The lemma is proved. ∎

Lemma 4. *Assume that $n > 1$ is a fixed integer, $\alpha \in (0,1)$, $M > 0$ and $\delta > 0$ are fixed constants. Then there exist constants $C > 0$ and $x_0 > 0$ such that for $p = 2$ and $p = 3$ we have*

$$\delta\left(x^{1/n} - \sum_{i=1}^{p} a_i^{1/n}\right) \geqslant C\left(x^{\alpha/n} - \sum_{i=1}^{p} a_i^{\alpha/n}\right) + Mx^{\alpha/n} \tag{3.6}$$

for every $x \geqslant x_0$ and $a_i \geqslant 0$ such that

$$\sum_{i=1}^{p} a_i \leqslant p^{1-n} \cdot x.$$

Proof. The left-hand side of (3.6) is always nonnegative by the Jensen inequality for the concave function $t^{1/n}$. Note that if $a_i = p^{-n}x$ for all $i = 1, \ldots, p$, then $x^{\alpha/n} - \sum_{i=1}^{p} a_i^{\alpha/n} = x^{\alpha/n}(1 - p^{1-\alpha}) < 0$. Fix C such that $C(2^{1-\alpha} - 1) > M$.

Then we may fix $\varepsilon > 0$ such that whenever $|a_i/x - p^{-n}| < \varepsilon$ for all $i = 1, \ldots, p$, the right-hand side of (3.6) is nonpositive and therefore (3.6) holds in this case.

By Proposition 1, otherwise there exists $\varepsilon_1 > 0$ such that left-hand side of (3.6) is not less than $\varepsilon_1 x^{1/n}$. It implies that (3.6) holds in this case for large enough x. ∎

Corollary 1. *Assume that $n > 1$ is a fixed integer, $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}_{>0}$ is a function satisfying $f(x) \geqslant \delta x^{1/n}$ for all $x \geqslant 0$, and*

$$f(x) \leqslant \max_{a_1+\cdots+a_p \leqslant x/p^{n-1}} \sum f(a_i) + Mx^{\alpha/n} \quad \forall x \geqslant x_0 \tag{3.7}$$

for $p \in \{2, 3\}$ and certain constants $x_0 > 0$, $\alpha \in (0,1)$, $M > 0$. Then there exist $C > 0$ and $x_1 > 0$ such that the function $\tilde{f}(x) := f(x) + Cx^{\alpha/n} - \delta x^{1/n}$ satisfies

$$\tilde{f}(x) \leqslant \max_{a_1+\cdots+a_p \leqslant x/p^{n-1}} \sum \tilde{f}(a_i) \quad \forall x \geqslant x_1. \tag{3.8}$$

Proof. Inequality (3.8) is obtained by subtracting (3.6) from (3.7). ∎

Now we give a recursive estimate for the maximal possible list chromatic number for an n-uniform hypergraph with prescribed number of edges.

LEMMA 5. *There exists a constant $M > 0$ such that for $p \in \{2, 3\}$ and all nonnegative integers N we have*

$$f_c(N) \leqslant \max_{a_1+\cdots+a_p \leqslant N/p^{n-1}} \sum_{i=1}^{p} f_c(a_i) + M(f_c(a_i))^{2/3}.$$

Proof. Let $H = (V, E)$ be an n-uniform hypergraph with $|E| = N$. Assume that any vertex $v \in V$ edge has a list $L(v)$ consisting of $\sum_{i=1}^{p} f_c(a_i) + c_i$ admissible colors, where

$$c_i := \lfloor M(f_c(a_i))^{2/3} \rfloor.$$

It suffices to prove that H has a proper list coloring with these lists.

As in the proof of Lemma 1, we partition V onto disjoints subsets V_i so that the corresponding induced subgraphs $H_i = (V_i, E_i)$ of H satisfy $\sum |E_i| \leqslant N/p^{n-1}$. Denote $a_i = |E_i|$.

For any color α choose $\xi(\alpha) \in \{1, \ldots, p\}$ independently at random with probability of $\{\xi(\alpha) = i\}$ proportional to $f_c(a_i) + c_i$. Call an edge $e \in E$ *nice* if it either contains the vertices from different V_i's, or $e \in E_i$ and $|L(v) \cap \xi^{-1}(i)| \geqslant f_c(a_i)$ for all n vertices $v \in e$. Due to Proposition 2 the probability that an edge $e \in E_i$ is not nice does not exceed

$$n \exp\left(-\frac{c_i^2}{2(f_c(a_i) + c_i)}\right)$$

(the multiple n comes from the number of vertices in e and applying the union bound).

If we permanently denote $f_c(a) = x$ for nonnegative integer a, $y = \lfloor Mx^{2/3} \rfloor$, then using the lower bound (3.1) and assuming $M > 100$ we conclude that

$$\frac{y^2}{2(y+x)} \geqslant \frac{M^2 x^{4/3}}{10 \max(x, Mx^{2/3})} = \frac{1}{10} \min\left(M^2 x^{1/3}, Mx^{2/3}\right) \geqslant \frac{Mx^{1/3}}{10} \geqslant \frac{M\delta^{1/3} a^{1/(3n)}}{10},$$

and

$$a \exp\left(-\frac{y^2}{2(x+y)}\right) < 1/n$$

for all $a = 0, 1, \ldots$ provided that the constant M is chosen large enough.

Fix such a value of M, then

$$n \sum_{i=1}^{p} a_i \exp\left(-\frac{c_i^2}{2(f_c(a_i) + c_i)}\right) < 1$$

and with positive probability all edges are nice. This allows us to properly color each H_i using the colors only from $\xi^{-1}(i)$ and get a proper coloring of H. ∎

Now Lemmas 3 and 5 for $p = 2$ yield $f_c(x) = O(x^{1/n})$. Therefore f_c satisfies the conditions of Corollary 1 for $\alpha = 2/3$ and certain $M > 0$ (and $x_0 = 1$). The corresponding function $\tilde{f}_c$ satisfies the conditions of Theorem 1, hence $f_c(x)/x^{1/n}$ has a finite limit and Theorem 3 is proved.

4. Further questions.

- First, recall that the Erdős conjecture is still open in the case $n = 3$. The survey and the best current lower bound are given in [3].
- A hypergraph is called *simple* if every pair of edges shares at most 1 vertex. Let $s(n,r)$ be the minimal number of edges in a simple n-graph which has no proper r-coloring. It is known [5] that for a fixed n one has

$$cr^{2n-2} \ln r \leqslant s(n,r) \leqslant Cr^{2n-2} \ln r.$$

  Unfortunately, we cannot show regularity of $s(n,r)$.
- Also it is natural to ask if $m(n,r)$ is regular on the first variable, i.e., does

$$\lim_{n\to\infty} \frac{m(n+1,r)}{m(n,r)} = r?$$

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