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Coloring general Kneser graphs and hypergraphs via high-discrepancy hypergraphs



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ABSTRACT

We suggest a new method for coloring generalized Kneser graphs based on hypergraphs with high discrepancy and a small number of edges. The main result provides a proper coloring of $K(n, n/2 - t, s)$ in $(4 + o(1))(s + t)^2$ colors, which is produced by Hadamard matrices. Also, we show that for colorings by independent set of a natural type, this result is the best possible up to a multiplicative constant.

Our method extends to Kneser hypergraphs as well.

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1. Introduction

Let $K(n, k, s)$ be the generalized Kneser graph, i.e. the graph with the vertex set $\binom{[n]}{k}$ and the edges connecting all pairs of vertices with intersection smaller than s , where $[n] = \{1, \dots, n\}$. Denote by $J(n, k, s)$ the generalized Johnson graph, i.e. the graph with the same vertex set $\binom{[n]}{k}$ and the edges connecting all pairs of vertices with intersection exactly s .

These graphs are quite popular objects in combinatorics. The chromatic number of the generalized Kneser graph was studied by Frankl and Füredi [9,10] for fixed k and s . Diameters of $K(n, k, s)$ and $J(n, k, s)$ are computed in [7] and [1] respectively.

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Bobu and Kupriyanov [6] studied the chromatic number of $J(n, n/2, s)$ for small values of s . They got the following results.

Theorem 1. For every $s < n/2$ we have

$$s + 2 \leq \chi \left[J \left(n, \frac{n}{2}, s \right) \right] \leq 2 \binom{2s + 1}{s + 1}.$$

Also, if $s \leq s' \sqrt{n}$ then for some $c = c(s')$

$$\chi \left[J \left(n, \frac{n}{2}, s \right) \right] \leq cn.$$

The lower bound immediately follows from the classical result of Lovász [17] on the Kneser graphs. It turns out that the upper bound is a particular case of our Lemma 2 with $t = 0$ and the complete hypergraph

$$H = \left([2s + 1], \binom{[2s + 1]}{s} \right).$$

More results on chromatic and clique-chromatic numbers of generalized Johnson graphs can be found in [22,23].

In this paper we improve the upper bound in Theorem 1 to a quadratic function in s for $s = O(\sqrt{n})$ using high-discrepancy hypergraphs. For $s \geq \sqrt{n}$ we have that the chromatic number grows as $n \cdot e^{(2+o(1))s^2/n}$ which simplifies to $e^{(2+o(1))s^2/n}$ when $n \ln n = o(s^2)$. The latter again corresponds to discrepancy results (see Section 4).

For every subset $A \subset [n]$ size of at least s we define the corresponding independent set in $K(n, k, s)$ (usually it is called Frankl's set):

$$I_A := \left\{ v \in \binom{[n]}{k} \mid |v \cap A| \geq \frac{|A| + s}{2} \right\}.$$

The complete intersection theorem of Ahlswede–Khachatryan [2] states that the independence number of $K(n, k, s)$ is always realized on some I_A (they give precise formulation what I_A is the biggest depending on n, k and s).

Now define the F -chromatic number as the chromatic number, that uses only coloring by Frankl's sets. Obviously, $\chi_F \geq \chi$ and since $J(n, n/2, s)$ is a subgraph of $K(n, n/2, s + 1)$,

$$\chi \left[J \left(n, \frac{n}{2}, s \right) \right] \leq \chi \left[K \left(n, \frac{n}{2}, s + 1 \right) \right].$$

Our first main theorem is the following.

Theorem 2. The following hold:

(i) if $s \leq \sqrt{n}/2$ then

$$\chi_F \left[K \left(n, \frac{n}{2}, s \right) \right] = \Theta(s^2);$$

(ii) if $\sqrt{n}/2 \leq s \leq O(\sqrt{n \ln n})$ then there is a constant $c > 0$ such that

$$n \cdot e^{cs^2/n} \leq \chi_F \left[K \left(n, \frac{n}{2}, s \right) \right] \leq n \cdot e^{(2+o(1))s^2/n};$$

(iii) if $\sqrt{n \ln n} \ll s$ then the chromatic number satisfies

$$\chi \left[K \left(n, \frac{n}{2}, s \right) \right] = e^{(2+o(1))s^2/n}.$$

The second main result is about generalized Kneser hypergraphs. Define

$$KH(n, r, k, s) = \left(\binom{[n]}{k}, \{ \{v_1, \dots, v_r\} \mid |v_i \cap v_j| < s \text{ for each } 1 \leq i < j \leq r \} \right).$$

Theorem 3. Let $n \geq m > (r(r-1)(s-1) + rt)^2$, and assume that there is a Hadamard matrix of size m . Then

$$\chi \left[KH \left(n, r, \frac{n}{r} - t, s \right) \right] \leq 2m.$$

The paper is organized as follows. In Section 2 we prove Theorem 5, which implies the lower bounds in Theorem 2(i) and (ii). In Section 3 we prove the upper bound in Theorem 2(i). In Section 4 we prove Theorem 2(iii) and the upper bound in Theorem 2(ii). In Section 5 we prove Theorem 3. And finally in Section 6, we give a geometric application, and we close the paper with some concluding remarks and open questions.

2. Lower bound for Theorem 2(i) and (ii)

The *discrepancy* of a hypergraph 2-coloring is the maximum over all edges of the difference between the number of vertices of the two colors in the edge. The *discrepancy* of a hypergraph H is the minimum discrepancy of among all 2-colorings of this hypergraph; we denote it by $\text{disc}(H)$.

Lemma 1. Let $t \leq n/2$ and $s \leq n/2 - t$, and consider a proper F -coloring of $K(n, n/2 - t, s)$. Let the family $\{A_i\}_{i=0}^q$ generate the Frankl's sets used for the proper F -coloring. Set $V = \cup_{i=0}^q A_i$ and $E = \{A_i\}_{i=0}^q$. Then the hypergraph $H = (V, E)$ has discrepancy at least s .

Proof. Consider an arbitrary 2-coloring of $[n]$. Obviously, there is a monochromatic (say, red) vertex v of $K(n, n/2 - t, s)$. Since v is contained in one of the Frankl's sets, there is an edge $A \in E$ containing at least $(|A| + s)/2$ red elements from $[n]$, so $\text{disc}(H) \geq s$. \square

We need the following theorem from [21] (it is also written in [5]).

Theorem 4. Let $H = (V, E)$ be a hypergraph. Then

$$\text{disc}(H) \leq 12\sqrt{|E|}.$$

By Lemma 1, a proper F -coloring of $K(n, n/2 - t, s)$ with q colors gives us a hypergraph H with q edges such that $\text{disc}(H) \geq s$. Then by Theorem 4

$$12\sqrt{q} \geq \text{disc}(H) \geq s \quad \text{so} \quad q \geq \frac{1}{144}s^2.$$

Thus, we proved the lower bound in Theorem 2(i) and a bit more.

Theorem 5. For every $t \leq n/2$ and $s \leq n/2 - t$ we have

$$\chi_F \left[K \left(n, \frac{n}{2} - t, s \right) \right] \geq \frac{1}{144}s^2.$$

The last displayed inequality in Section 12.2 in [4] states that there is a constant M such that for every hypergraph $H = (V, E)$ with $|V| < |E|$ the following holds

$$\text{disc}(H) \leq M \sqrt{|V| \ln \frac{|E|}{|V|}}.$$

Now we can prove the lower bound in Theorem 2(ii). If there is an F -coloring with q colors then

$$M \sqrt{n \ln \frac{q}{n}} \geq \text{disc}(H) \geq s,$$

and hence

$$q \leq n \cdot e^{s^2 M^{-2} n^{-1}}.$$

3. Upper bound in Theorem 2(i)

The *t*-shifted discrepancy of a 2-coloring is the maximum over all edges *e* of the quantity

$$|\text{blue}(e) - \text{red}(e) + t|,$$

where *blue*(*e*), *red*(*e*) mean the number of blue and red vertices in *e*. The *t*-shifted discrepancy of a hypergraph *H* is the minimum discrepancy of among all 2-colorings of this hypergraph.

Lemma 2. *Let $H = (V, E)$ be a hypergraph with *t*-shifted discrepancy at least $s + t$, and $|V| \leq n$. Then*

$$\chi_F \left[K \left(n, \frac{n}{2} - t, s \right) \right] \leq 2|E|.$$

Proof. Embed *H* into $[n]$. For every edge $e \in E$ define colors 1_e and 2_e as follows:

$$1_e := \left\{ A \in V \left(K \left(n, \frac{n}{2} - t, s \right) \right) \mid |A \cap e| \geq \frac{|e| + s}{2} \right\};$$

$$2_e := \left\{ A \in V \left(K \left(n, \frac{n}{2} - t, s \right) \right) \mid |A \cap \bar{e}| \geq \frac{|\bar{e}| + s}{2} \right\}.$$

Vertices of the same color span an independent set (and it is a Frankl's set by definition), because every pair of vertices with the same color intersects by at least *s* points.

Every set $A \subset [n]$ of size $\frac{n}{2} - t$ gives a 2-coloring of *H* by setting blue color to $V(H) \cap A$ and red color to $V(H) \setminus A$. By the condition on the *t*-shifted discrepancy, there is a hyperedge $e \in H$ such that

$$||A \cap e| - |\bar{A} \cap e| + t| \geq s + t.$$

Since $|A \cap e| + |\bar{A} \cap e| = |e|$, we have

$$|A \cap e| \geq \frac{|e| + s}{2} \quad \text{or} \quad |A \cap e| \leq \frac{|e| - s - 2t}{2}.$$

It means that $A \in 1_e$ or $A \in 2_e$ respectively, because $|A \cap \bar{e}| \geq \frac{|\bar{e}| + s}{2}$ is equivalent to $|A \cap e| \leq \frac{|e| - (s + 2t)}{2}$.

To summarize: all the vertices are colored, and every color class is a Frankl's set, i.e. our coloring is a proper *F*-coloring. □

Theorem 6. *Let $n \geq m > 4(s + t)^2$, and assume that there is a Hadamard matrix of size *m*. Then*

$$\chi_F \left[K \left(n, \frac{n}{2} - t, s \right) \right] \leq 2m.$$

Proof. It is well-known that a Hadamard matrix of size *m* produces a hypergraph with discrepancy at least $\sqrt{m}/2$. We repeat the proof of this from [5] and show that for any *t* it has *t*-shifted discrepancy at least $\sqrt{m} - 1/2$.

Let $H = \{h_{ij}\}$ be a Hadamard matrix of order *m* with first row and first column all ones. Any Hadamard matrix can be so “normalized” by multiplying appropriate rows and columns by -1 . Let $v = (v_1, \dots, v_m)$, $v_i = \pm 1$. Then

$$Hv = v_1c_1 + \dots + v_m c_m,$$

where c_i denotes the *i*th column vector of *H*. Writing $Hv = (L_1, \dots, L_m)$ and letting $|c|$ denote the usual Euclidean norm,

$$L_1^2 + \dots + L_m^2 = |Hv|^2 = v_1^2|c_1|^2 + \dots + v_m^2|c_m|^2 = m + \dots + m = m^2,$$

since the c_i 's are mutually orthogonal. Note also that

$$L_1 + \dots + L_m = \sum_{i,j=1}^m v_j h_{ij} = \sum_{j=1}^m v_j \sum_{i=1}^m h_{ij} = m v_1 = \pm m.$$

Let J be the all ones matrix of order m . Set $\lambda = v_1 + \dots + v_m$ so that $Jv = (\lambda, \dots, \lambda)$ and

$$\frac{H+J}{2}v = \left(\frac{L_1+\lambda}{2}, \dots, \frac{L_m+\lambda}{2}\right).$$

Let $x = \lambda - 2t$ so

$$\sum_{i=1}^m (L_i+x)^2 = \sum_{i=1}^m L_i^2 + 2x \sum_{i=1}^m L_i + mx^2 = m^2 \pm 2mx + mx^2 \geq m(m-1). \tag{1}$$

This implies that for some i we have

$$\left|\frac{L_i+\lambda}{2} - t\right| \geq \frac{\sqrt{m-1}}{2}.$$

Lemma 2 finishes the proof. \square

It is worth noting that a random approach gives the same (up to a constant factor) bound, see [5]. Theorem 6 implies the upper bound in Theorem 2(i).

4. Proof of Theorem 2(iii) and the upper bound in Theorem 2(ii)

Frankl and Wilson proved [14] the following theorem.

Theorem 7. *Let $s \leq n/4$ and $n/2 - s$ be a primary (prime or a power of a prime) number. Then*

$$\alpha \left[J \left(n, \frac{n}{2}, s \right) \right] \leq \binom{n}{\frac{n}{2} - s - 1}.$$

Now we are working in the setup $s^2 = \Omega(n)$. First, suppose that $n/2 - s$ is a primary number. Then the lower bound on the chromatic number of $K := K(n, n/2, s)$ becomes non-trivial:

$$\frac{\binom{n}{n/2}}{\binom{n}{n/2-s}} = \frac{(n/2+s) \cdot \dots \cdot (n/2+1)}{(n/2) \cdot \dots \cdot (n/2-s+1)} = \frac{\prod_{i=1}^s (1 + \frac{i}{n/2})}{\prod_{i=0}^{s-1} (1 - \frac{i}{n/2})} = e^{(2+o(1))s^2/n}.$$

The prime numbers are dense, so again

$$\chi \left[K \left(n, \frac{n}{2}, s+1 \right) \right] \geq \chi \left[J \left(n, \frac{n}{2}, s \right) \right] \geq \frac{|V(J)|}{\alpha(J)} \geq e^{(2+o(1))s^2/n}. \tag{2}$$

To show that the bound is near to optimal, recall the theorem of Lovász on fractional covers [16]. Let τ be the minimal number of edges required to cover all vertices of H and let τ^* be the minimal sum of weights on edges required to cover all the vertices of H in such a way that the sum of weights over edges containing every vertex is at least 1.

Theorem 8. *Let $H = (V, E)$ be a hypergraph, d be the maximum edge size of H . Then*

$$\tau(H) \leq (1 + \ln d)\tau^*.$$

Now we prove that we have equality in (2). Let $A \subset [n]$ be a set that realizes $\max_A |I_A|$. We shall apply Theorem 8 to the hypergraph

$$H := \left(\binom{[n]}{\frac{n}{2}}, \{ \pi(I_A) \mid \pi \in S_n \} \right),$$

where S_n is the permutation group over $[n]$. A collection of hyperedges covering $V(H)$ provides a proper F -coloring of K , so

$$\tau(H) \geq \chi_F(K) \geq \chi(K).$$

By vertex transitivity we have

$$\tau^*(H) = \frac{|V(H)|}{d(H)} = \frac{|V(K)|}{\alpha(K)}.$$

Putting this all together

$$\chi_F(K) \leq \tau(H) \leq \tau^*(1 + \ln d(H)) \leq \frac{|V(K)|}{\alpha(K)}(1 + \ln d(H)) \leq ne^{(2+\alpha(1))s^2/n}. \tag{3}$$

Because of (2) for the case $n \ln n = o(s^2)$ we have

$$\chi \left[K \left(n, \frac{n}{2}, s \right) \right] = e^{(2+\alpha(1))s^2/n}.$$

5. Hypergraph case

In this section we are going to extend our methods to generalized Kneser hypergraphs.

It turns out that we can repeat the arguments for graphs in this case. The *w-shifted r-centered discrepancy* of a 2-coloring is the maximum over all edges of the quantity

$$|(r - 1)blue(e) - red(e) + w|,$$

where *blue(e)* and *red(e)* mean the number of blue and red vertices in *e*. The *r-centered discrepancy* of a hypergraph *H* is the minimum discrepancy of among all 2-colorings of this hypergraph.

Lemma 3. *Let $H = (V, E)$ be a hypergraph with $tr/2$ -shifted r -centered discrepancy $x > \frac{r(r-1)(s-1)+rt}{2}$, and $|V| \leq n$. Then*

$$\chi \left[KH \left(n, r, \frac{n}{r} - t, s \right) \right] \leq 2|E|.$$

Proof. Embed *H* into $[n]$. For every edge $e \in E$ define colors 1_e and 2_e as follows:

$$1_e := \left\{ A \in V \left(K \left(n, r, \frac{n}{r} - t, s \right) \right) \mid |A \cap e| > \frac{|e|}{r} + \frac{(r - 1)(s - 1)}{2} \right\};$$

$$2_e := \left\{ A \in V \left(K \left(n, r, \frac{n}{r} - t, s \right) \right) \mid |A \cap \bar{e}| > \frac{|\bar{e}|}{r} + \frac{(r - 1)(s - 1)}{2} \right\}.$$

First, let us show that every vertex has a color. Every set $A \subset [n]$ of size $\frac{n}{r} - t$ gives a 2-coloring of *H* by setting blue color to $V(H) \cap A$ and red color to $V(H) \setminus A$. By the condition on the $tr/2$ -shifted r -centered discrepancy, there is a hyperedge $e \in H$ such that

$$\left| (r - 1)blue(e) - red(e) + \frac{tr}{2} \right| \geq x > \frac{r(r - 1)(s - 1)}{2} + \frac{tr}{2}.$$

Using $blue(e) = |A \cap e|$ and $red(e) = |e| - |A \cap e|$ we have either

$$-|e| + r|A \cap e| + \frac{tr}{2} > \frac{r(r - 1)(s - 1)}{2} + \frac{tr}{2}$$

or

$$-|e| + r|A \cap e| + \frac{tr}{2} < -\frac{r(r - 1)(s - 1)}{2} - \frac{tr}{2}.$$

In the first case

$$|A \cap e| > \frac{|e|}{r} + \frac{(r - 1)(s - 1)}{2},$$

i.e. *A* is colored by 1_e . In the second case

$$|A \cap e| < \frac{|e|}{r} - \frac{(r - 1)(s - 1)}{2} - t.$$

Using $|A \cap \bar{e}| + |A \cap e| = \frac{n}{r} - t$ we have

$$|A \cap \bar{e}| > \frac{n}{r} - t - \frac{|e|}{r} + \frac{(r-1)(s-1)}{2} + t = \frac{|\bar{e}|}{r} + \frac{(r-1)(s-1)}{2},$$

i.e. A is colored by $2e$.

Suppose that there is a monochromatic edge $\{v_1, \dots, v_r\}$. Then there is an edge or the complement of an edge (denote it by e) such that for $i = 1, \dots, r$ the following holds

$$|v_i \cap e| > \frac{|e|}{r} + \frac{(r-1)(s-1)}{2}.$$

This implies

$$\sum_{i=1}^r |v_i \cap e| > |e| + r \frac{(r-1)(s-1)}{2}.$$

From the other hand, since $|v_i \cap v_j| < s$ we have

$$|e| \geq \sum_{i=1}^r |v_i \cap e| - \frac{r(r-1)}{2}(s-1),$$

a contradiction. \square

The proof of [Theorem 3](#) is similar to the proof of [Theorem 6](#) with the replacement of [Lemma 2](#) with [Lemma 3](#). Since the inequality (1) holds for every x and all the sets from Hadamard matrix have the size $m/2$ it works for centered and shifted discrepancy.

6. A geometric application

First, we need some additional definitions. Let

$$V_k := \{v \in \{0, \pm 1\}^n \mid |v| = \sqrt{k}\};$$

$$V_{k,l} := \{v \in \{0, \pm 1\}^n \mid v \text{ has exactly } k \text{ '1' and exactly } l \text{ '-1'}\}.$$

Furthermore, we define

$$K(n, k, s) := (V_k, \{(v_1, v_2) \mid (v_1, v_2) < s\}); \quad K(n, k, l, s) := (V_{k,l}, \{(v_1, v_2) \mid (v_1, v_2) < s\});$$

$$J(n, k, s) := (V_k, \{(v_1, v_2) \mid (v_1, v_2) = s\}); \quad J(n, k, l, s) := (V_{k,l}, \{(v_1, v_2) \mid (v_1, v_2) = s\}),$$

where (v_1, v_2) is the standard inner product. Also, the *support* of a vector is the set $\text{supp}(v) \in [n]$ of its non-zero coordinates.

Obviously, there is a natural bijection between the subsets of $[n]$ and $\{0, 1\}$ -vectors in \mathbb{R}^n . If we fix the size of subsets, then the corresponding vectors lie on a sphere. It implies one-to-one correspondence between the scalar product and the Euclidean distance in this case. So $J(n, k, s)$ is a distance graph.

The independence number of such a graph was studied [[11–13](#)] by Frankl and Kupavskii. In [[12](#)] the authors find an explicit value of the independence number of $J(n, k, 1, -2)$. Paper [[11](#)] is devoted to the independence numbers of $K(n, k, s)$ with $n > n_0(k, s)$. Finally, the work [[13](#)] deals with the independence number of $J(n, k, l, -2l)$ for arbitrary n, k and l .

It is also worth noting that Cherkashin, Kulikov and Raigorodskii [[8](#)] improved lower bounds on the chromatic numbers of small-dimensional Euclidean spaces via chromatic numbers of $J(n, 3, 1)$.

In this setup s is the difference between the minimal scalar product and the maximum restricted scalar product; t is the difference between $n/2$ and the size of a vertex support. Using [Theorem 6](#) we have that the chromatic number is at most quadratic in the small parameters.

Corollary 1. *Let $n \geq m > 4(s + 2l + t)^2$, and assume that there is a Hadamard matrix of size m . Then*

$$\chi \left[K \left(n, \frac{n}{2} - l - t, l, -2l + s \right) \right] \leq 2m.$$

Proof. Note that if vertices v_1, v_2 are such that $|\text{supp}(v_1) \cap \text{supp}(v_2)| = q$, then the scalar product $\langle v_1, v_2 \rangle \geq q - 4l$. So for $q \geq 2l + s$ the vertices are not adjacent. Hence

$$\chi \left[K \left(n, \frac{n}{2} - l - t, l, -2l + s \right) \right] \leq \chi \left[K \left(n, \frac{n}{2} - t, 2l + s \right) \right].$$

Using [Theorem 6](#) we are done. \square

7. Discussion

Improving the lower bound for general colorings. Recall that the exact value of the chromatic number of Kneser graph was determined by Lovász [17]. Then Alon, Frankl and Lovász [3] determined the chromatic number of Kneser hypergraphs, i.e. proved that

$$\chi(KH(n, r, k, 1)) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$

Since then several different proofs have appeared. One of the two main ways to prove the lower bound uses Borsuk–Ulam theorem (or its analogues), see [18–20,24] and the other computes the connectedness of a corresponding complex, see [3,15,17].

Recall that for k being almost n/r we have quadratic dependence on small parameters. Unfortunately, we are not able to improve the linear lower bound and this problem looks quite challenging.

Discrepancy. In Section 4 we showed that there is a close relation between F -chromatic number and the discrepancy theory. For instance, [Lemma 1](#) and relation (3) imply the existence of a hypergraph $H = (V, E)$ for every $|E| \geq |V|$ such that

$$\text{disc}(H) \geq (1 + o(1)) \sqrt{\frac{|V| \ln \frac{|E|}{|V|}}{2}},$$

which is optimal up to a constant.

It might be interesting to explore the concept of shifted and centered discrepancy in general.

Constant values of s . Obviously, $\chi[K(n, n/2, 1)] = J(n, n/2, 0) = \chi_F[K(n, n/2, 1)] = J(n, n/2, 0) = 2$, because the graph is a matching. Note that $\text{disc}(\{3\}, \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}) = 2$, so by [Lemma 2](#) we have the following observation

$$\chi_F \left[K \left(n, \frac{n}{2}, 2 \right) \right] \leq 6.$$

Also Fano’s plane has discrepancy 3, so $\chi_F \left[K \left(n, \frac{n}{2}, 3 \right) \right] \leq 14$, but this seems not to be optimal. On the other hand, [Theorem 1](#) gives $3 \leq \chi_F \left[K \left(n, \frac{n}{2}, 2 \right) \right]$ and $4 \leq \chi_F \left[K \left(n, \frac{n}{2}, 3 \right) \right]$. Finding the exact values of the chromatic numbers for constant s is also of some interest.

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