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Boundary-layer approach to high-frequency diffraction by a jump of curvature

Ekaterina A. Zlobina [b,*], Aleksei P. Kiselev [a,b,c]

[a] St. Petersburg Department of V.A. Steklov Institute of Mathematics of the Russian Academy of Sciences, Fontanka Emb. 27, St. Petersburg 191023, Russia
[b] St. Petersburg State University, Universitetskaya Emb. 7-9, St. Petersburg, 199034, Russia
[c] Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences, Vasilievsky Ostrov Bolshoy Prospect 61, St. Petersburg 199178, Russia

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ABSTRACT

A systematic boundary-layer approach is for the first time applied to diffraction of a high-frequency plane wave by a contour with a jump of curvature. Assuming that the incident wave is non-tangent, we present a detailed description of the outgoing wavefield within a boundary layer surrounding the point of non-smoothness of the contour. This allows us to describe the wavefield within a transition zone surrounding the limit ray in terms of the parabolic cylinder function D_{-3} which has not been previously encountered in high-frequency diffraction problems.

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1. Introduction

The structure of high-frequency wavefields in diffraction problems is described by the Geometrical Theory of Diffraction (GTD) clearly articulated by J.B. Keller [1]. Crude versions of GTD present a wavefield as a sum of contributions of rays reflected from smooth parts of the boundary and rays associated with diffraction from points of its non-smoothness. Formulas for diffracted waves should be found in the course of consideration of related simple model problems typically allowing separation of variables. The problem of diffraction by a contour with a jump of curvature has for a long time attracted attention of researchers (see, e.g., [2–8]) not only by its possible applications, but also (and mainly) because no simple model problem to describe diffraction by such a singularity is available. So far, a treatment of this problem has been based on the Kirchhoff method (see, e.g., [9]), which consists essentially in application the Green formula where the values of wavefields and their normal derivatives on the contour are formally replaced by leading-order terms of ray expressions.

The pioneering investigation of high-frequency diffraction by a jump of curvature was undertaken by A. V. Popov [2] who addressed a plane wave incident along a planar (straight) boundary (with the Neumann condition), passing into a parabola at its apex. Later, several problems with tangential incidence were explored by N. Ya. Kirpichnikova,

* Corresponding author.
E-mail addresses: ezlobina2@yandex.ru (E.A. Zlobina), kiselev@pdmi.ras.ru (A.P. Kiselev).

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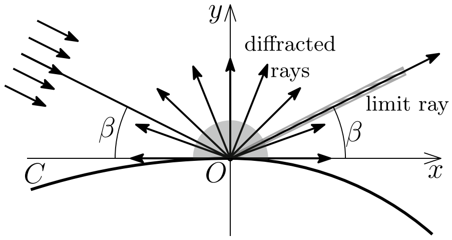

Fig. 1. Rays of diffracted and incident waves and boundary layers.

A. S. Kirpichnikova and V. B. Philippov [3–5]. They addressed diffraction of creeping and whispering gallery waves on a curved boundary with a jump in curvature, for either Dirichlet or Neumann boundary conditions. This analysis was generalized to an elastic media [5]. In the paper by L. Kaminetsky and J. B. Keller [6], diffraction was investigated in the case of a non-tangent incidence on a curvilinear boundary with an isolated point at which a curvature or its derivative jumps. For a variety of boundary conditions they derived expressions for the diffracted wave; however, they did not discuss the wavefield in the immediate proximity to the direction of specular reflection. A non-stationary approach to high-frequency diffraction by a contour with a jump of curvature, similar to the Kirchhoff method, has been developed (for ideal boundary conditions) by A. F. Filippov [7]. A. P. Kiselev and Z. M. Rogoff in [8] described a specific effect of impedance on outgoing wavefield. In all above research, expressions were found for a diffracted cylindrical wave, which agree with each other in the sense that their diffraction coefficients have third-order poles on the limit ray.

The aforementioned crude GTD, providing ray formulas for diffracted waves, fails in small vicinities of points of non-smoothness of the boundary and in narrow transition zones (such as penumbras) where phases of reflected and diffracted waves merge and the waves lose their individuality. J. B. Keller and R. N. Buchal [10] first pointed to the importance of describing wavefields in transition zones by the boundary layer approach. An extensive account of employment of the boundary layer techniques in high-frequency diffraction problems was given by V. M. Babič and N. Ya. Kirpičnikova [11]. However, to the best of our knowledge, the boundary-layer techniques has never been systematically applied to diffraction by a jump of curvature, as we believe, because of arising analytical difficulties.

In the present paper, we apply the boundary layer approach (see, e.g., [11]) to high-frequency diffraction by a contour with a jump of curvature. First, we study the wavefield in a small neighborhood of the point of non-smoothness of the contour. We introduce there stretched coordinates and explicitly solve related non-standard singular boundary-value problems describing the "local" wavefield. On this basis, we find the respective "far-field" asymptotics of the "local" wavefield in the area where the distance from the singular point ρ is small[1] but $k\rho \to \infty$, with k standing for the wavenumber. The procedure allows an expression for diffracted wave, which agrees with earlier known results [2–8].

We use a kind of boundary layer description of the wavefield around the specularly reflected, or limit, ray. Matching it with an expression for wavefield near the singular point allows a representation of the wavefield in terms of the parabolic cylinder function D_{-3} which has not been earlier encountered in diffraction theory.

2. Formulation of problem

We assume the harmonic time dependence $e^{-i\omega t}$, where t is time, ω is circular frequency related to wavenumber k by $\omega/c = k$, and $c = const$ is wave speed. We put $c = 1$. Total wavefield $u = u^{\mathrm{i}} + u^{\mathrm{o}}$, where u^{i} and u^{o} are incident and outgoing wavefields,[2] respectively, is governed by the Helmholtz equation

$$\Delta u + k^2 u = 0, \tag{1}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian and we assume that the wavenumber k is large:

$$k \to \infty. \tag{2}$$

The wavefield u is considered in an infinite domain bounded by a contour C of which the curvature æ is smooth except for a point O where it has a jump (see Fig. 1). We parameterize C by its arc length s measured from O (see Fig. 3) and assume that its curvature $æ = æ(s)$ has the following form:

$$æ(s) = g(s) + hH(s). \tag{3}$$

Here, $g(s)$ is a smooth function, $g(0) = æ_0$, h is the magnitude of jump, and $H(s)$ is the Heaviside function

$$H(s) = \begin{cases} 0, & \text{if } s < 0; \\ 1, & \text{if } s \geq 0. \end{cases} \tag{4}$$

[1] To be precise, we compare ρ with a certain geometrical parameter specified in (5).

[2] Henceforth the outgoing wave satisfies the limiting absorption principle, which will be discussed in more detail in Section 4.

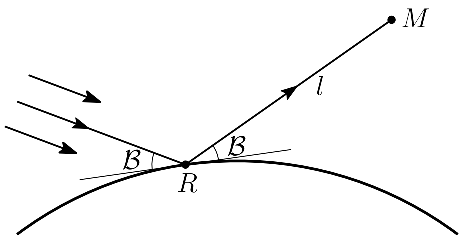

Fig. 2. Reflection from a smooth contour.

No assumption about the convexity of the contour C is made. Accordingly, æ(s) may be of any sign.

In order to non-dimensionalize subsequent relations, we introduce a characteristic geometrical parameter of the problem under consideration:

$$\varkappa = \max\{|æ_0|, |æ_0 + h|\}. \tag{5}$$

We assume that on the contour C the Dirichlet boundary condition holds, that is

$$u^{\mathrm{o}}\big|_C = -\, u^{\mathrm{i}}\big|_C\,. \tag{6}$$

Let the incident wave be a plane wave $u^{\mathrm{i}} = e^{ik(x\cos\beta - y\sin\beta)}$, where x and y are Cartesian coordinates with the origin at O and x-axis tangent to C, β is the grazing angle (see Fig. 1). We assume the incidence to be non-tangent, that is, $\beta > \varepsilon > 0$ with ε fixed and independent of k.

It is convenient to introduce linear functions $\mathrm{P}^{\pm}(x, y)$, which will be useful in description of the wavefield:

$$\mathrm{P}^{\pm}(x, y) = x\cos\beta \pm y\sin\beta. \tag{7}$$

Thus, the incident wave is

$$u^{\mathrm{i}} = e^{ik\mathrm{P}^{-}(x,y)}. \tag{8}$$

The function P^{+} is associated with phase of an outgoing plane wave.

The outgoing wavefield u^{o} satisfies the Helmholtz equation (1) and the boundary condition (6). According to the crude GTD (see, e.g., [1]), at some distance from the limit (specularly reflected) ray u^{o} is a sum of u^{r}, the wave specularly reflected from smooth parts of the contour, and u^{d}, the diffracted wave:

$$u^{\mathrm{o}} = u^{\mathrm{r}} + u^{\mathrm{d}}. \tag{9}$$

The latter is a cylindrical wave (see, e.g, [6])

$$u^{\mathrm{d}} = A(\phi, k)\frac{e^{ik\rho}}{\sqrt{k\rho}}\left(1 + o\left(1\right)\right), \quad k\rho \to \infty. \tag{10}$$

Here, $A(\phi, k)$ is a diffraction coefficient and ρ and ϕ are classical polar coordinates centered at O:

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad -\pi < \phi \le \pi. \tag{11}$$

For a plane wave incidence onto a smooth contour, asymptotics of reflected wave u^{r} is described by the well-known ray formula (see, e.g., [12])

$$u^{\mathrm{r}} = -\left(1 + \frac{2æl}{\sin\mathcal{B}}\right)^{-\frac{1}{2}} e^{ik\tau}\left(1 + O\left(\frac{1}{k\rho}\right)\right), \quad k\rho \to \infty. \tag{12}$$

Here, $\tau = \tau(M)$ stands for the value of eikonal at the observation point M, l is the distance from the respective point of specular reflection R to the point M, $\mathcal{B}$ is the grazing angle at R, and the value æ of curvature is taken at the point R (see Fig. 2).

In the case of a jump of curvature at O, expression (12) jumps at the limit ray. Therefore, a description of wavefield in its neighborhood requires a special consideration which we present in Section 5. Also, formula (12) shows that close to the point O a rough approximation for the reflected wave is

$$u^{\mathrm{r}} \approx w^{\mathrm{r}} = -e^{ik\mathrm{P}^{+}(x,y)}. \tag{13}$$

In Appendix A a more detailed approximation taking into account the curvature is derived

$$u^{\mathrm{r}} \approx w^{\mathrm{r}} + v^{\mathrm{r}}\left(1 + O(\varkappa\rho) + O\left(k\varkappa\rho^2(\phi - \beta)^2\right)\right), \tag{14}$$

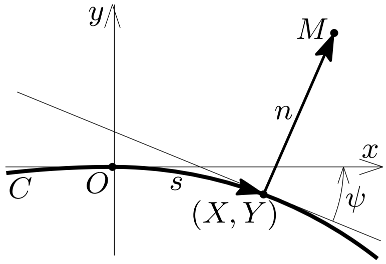

Fig. 3. Coordinates s and n.

where

$$v^{\mathrm{r}} = (æ_0 + hH(\beta - \phi))\left(\frac{y}{\sin^2\beta} - ik\frac{(x\sin\beta - y\cos\beta)^2}{\sin\beta}\right)e^{ik\mathrm{P}^+(x,y)}. \tag{15}$$

It can be seen from (14) and (15) that $|v^{\mathrm{r}}| \ll |w^{\mathrm{r}}|$ in the area where

$$\varkappa\rho \ll 1, \tag{16}$$

and

$$k\varkappa\rho^2(\phi - \beta)^2 \ll 1. \tag{17}$$

Conditions (16) and (17) hold true under the assumption

$$k\varkappa\rho^2 \ll 1. \tag{18}$$

It will be observed that the inequality (18) characterizes the size of a boundary layer surrounding point O (see Section 3), whereas condition (17) characterizes the width of the transition zone surrounding the limit ray (see Section 5).

In what follows, the wavefield will be investigated by the boundary layer approach in two areas: in a small neighborhood of point O and in the vicinity of the limit ray (gray-filled and gray-crosshatched in Fig. 1, respectively). Our consideration will be confined to the area described by inequality (16).

3. Boundary layer equations near point *O*

3.1. Coordinate systems

We describe the position of a point of observation M close to O with the help of orthogonal coordinates s and n, where n is the length of segment of perpendicular dropped from M onto the contour C (see Fig. 3). For point M positioned above the contour $n \geq 0$.

It is well known (see, e.g., [13]) that the Cartesian coordinates of the point M can be represented as follows

$$x = X(s) - n\sin\psi, \quad y = Y(s) + n\cos\psi. \tag{19}$$

Here, ψ is angle between the tangent to the contour C and positive direction of x-axis, $-\pi < \psi \leq \pi$, and the contour C is parametrically described by formulas

$$X(s) = \int_0^s \cos\psi(s)ds, \quad Y(s) = \int_0^s \sin\psi(s)ds. \tag{20}$$

The angle ψ is related to the curvature for $s \neq 0$ as follows

$$æ(s) = -\frac{d\psi(s)}{ds}. \tag{21}$$

Thus, for small s we have

$$\psi(s) = -\int_0^s æ(s)ds = -\left(æ_0 s + hs_+\right)\left(1 + o(1)\right), \tag{22}$$

where

$$s_+^\lambda = s^\lambda H(s), \qquad s_+ := s_+^1, \tag{23}$$

and H is the Heaviside function (4). Employing Eqs. (19) and (20) we obtain

$$\begin{cases} x = s + æ_0\, ns + hns_+ + o\left(\varkappa\left(n^2+s^2\right)\right), \\ y = n - \dfrac{æ_0 s^2}{2} - \dfrac{hs_+^2}{2} + o\left(\varkappa\left(n^2+s^2\right)\right). \end{cases} \tag{24}$$

In a small neighborhood of the point O we introduce the standard stretched coordinates S and N (see, e.g., [14,15])

$$S = ks, \quad N = kn \tag{25}$$

and expand the Cartesian coordinates x and y in inverse powers of the large parameter k:

$$\begin{cases} x = \dfrac{S}{k} + \dfrac{æ_0 NS}{k^2} + \dfrac{hNS_+}{k^2} + o\left(\dfrac{\varkappa(S^2+N^2)}{k^2}\right), \\ y = \dfrac{N}{k} - \dfrac{æ_0 S^2}{2k^2} - \dfrac{hS_+^2}{2k^2} + o\left(\dfrac{\varkappa(S^2+N^2)}{k^2}\right). \end{cases} \tag{26}$$

3.2. Incident wave in stretched coordinates

With the help of (26), the phases of incident and outgoing plane waves (7) become

$$\begin{aligned} k\mathrm{P}^{\pm}(x,y) &= k(x\cos\beta \pm y\sin\beta) \\ &= \mathrm{P}^{\pm}(S,N) + \frac{1}{k}\left[æ_0\left(\mp\frac{\sin\beta}{2}S^2 + \cos\beta\, NS\right)\right. \\ &\quad \left. + h\left(\mp\frac{\sin\beta}{2}S_+^2 + \cos\beta\, NS_+\right)\right] + o\left(\frac{\varkappa(S^2+N^2)}{k}\right), \quad k\to\infty. \end{aligned} \tag{27}$$

Notice that

$$\mathrm{P}^{\pm}(S,N) = k\mathrm{P}^{\pm}(s,n). \tag{28}$$

Now, the incident wave u^{i} in the vicinity of the point O is represented as follows:

$$u^{\mathrm{i}} = e^{ik\mathrm{P}^-(x,y)} = u_0^{\mathrm{i}} + \frac{1}{k}\left(æ_0 u_1^{\mathrm{i}} + hu_1^{\mathrm{i}h}\right) + \cdots, \quad k\to\infty, \tag{29}$$

where the functions

$$u_0^{\mathrm{i}} = e^{i\mathrm{P}^-(S,N)}, \qquad u_1^{\mathrm{i}} = i\left(\frac{\sin\beta}{2}S^2 + \cos\beta\, NS\right)e^{i\mathrm{P}^-(S,N)}, \qquad u_1^{\mathrm{i}h} = i\left(\frac{\sin\beta}{2}S_+^2 + \cos\beta\, NS_+\right)e^{i\mathrm{P}^-(S,N)} \tag{30}$$

are independent of h.

The asymptotics for the reflected wave u^{r} (14) has a similar form:

$$u^{\mathrm{r}} = w_0^{\mathrm{r}} + \frac{1}{k}\left(æ_0(w_1^{\mathrm{r}} + v_1^{\mathrm{r}}) + h(w_1^{\mathrm{r}h} + v_1^{\mathrm{r}h})\right) + \cdots, \quad k\to\infty, \tag{31}$$

with

$$w_0^{\mathrm{r}} = -e^{i\mathrm{P}^+(S,N)}, \qquad w_1^{\mathrm{r}} = i\left(\frac{\sin\beta}{2}S^2 - \cos\beta\, NS\right)e^{i\mathrm{P}^+(S,N)}, \qquad w_1^{\mathrm{r}h} = i\left(\frac{\sin\beta}{2}S_+^2 - \cos\beta\, NS_+\right)e^{i\mathrm{P}^+(S,N)}, \tag{32}$$

$$\begin{aligned} v_1^{\mathrm{r}} &= \left(\frac{N}{\sin^2\beta} - i\frac{(S\sin\beta - N\cos\beta)^2}{\sin\beta}\right)e^{i\mathrm{P}^+(S,N)}, \\ v_1^{\mathrm{r}h} &= H(S\sin\beta - N\cos\beta)\left(\frac{N}{\sin^2\beta} - i\frac{(S\sin\beta - N\cos\beta)^2}{\sin\beta}\right)e^{i\mathrm{P}^+(S,N)}. \end{aligned} \tag{33}$$

3.3. Helmholtz equation in stretched coordinates

In coordinates s and n the Helmholtz operator can be written as (see, e.g., [13]):

$$\Delta + k^2 = \frac{1}{1+næ}\frac{\partial}{\partial s}\left(\frac{1}{1+næ}\frac{\partial}{\partial s}\right) + \frac{1}{1+næ}\frac{\partial}{\partial n}\left((1+næ)\frac{\partial}{\partial n}\right) + k^2. \tag{34}$$

Expanding the right-hand side of formula (34) in powers of n and passing to the coordinates S and N, we get

$$\Delta + k^2 = k^2\left(L_0 + \frac{1}{k}\left(æ_0 L_1 + hL_1^h\right) + \cdots\right), \quad k\to\infty, \tag{35}$$

where we introduce notations:

$$L_0 = \frac{\partial^2}{\partial N^2} + \frac{\partial^2}{\partial S^2} + 1, \qquad L_1 = -2N\frac{\partial^2}{\partial S^2} + \frac{\partial}{\partial N}, \qquad L_1^h = -2NH(S)\frac{\partial^2}{\partial S^2} + H(S)\frac{\partial}{\partial N} - N\delta(S)\frac{\partial}{\partial S}. \tag{36}$$

Here, H is the Heaviside function (4) and δ is the Dirac delta function.

3.4. Boundary value problems in stretched coordinates

We seek the solution of the boundary value problem (1), (6) in the form

$$u^{\mathrm{o}} = U_0^{\mathrm{o}} + \frac{1}{k}\left(æ_0 U_1^{\mathrm{o}} + hU_1^{\mathrm{o}h}\right) + \cdots, \quad k \to \infty. \tag{37}$$

Aiming at asymptotic description of the effect of a curvature jump on the outgoing wavefield, we address solely the term linear in h. Substituting (37) into (1) and (6) and equating to zero coefficients of powers of k, we arrive at boundary value problems

$$\begin{cases} L_0 U_0^{\mathrm{o}} = 0, & \text{(a)} \\ U_0^{\mathrm{o}}\big|_{N=0} = -\, u_0^{\mathrm{i}}\big|_{N=0}, & \text{(b)} \end{cases} \tag{38}$$

and

$$\begin{cases} L_0 U_1^{\mathrm{o}h} + L_1^h U_0^{\mathrm{o}} = 0, & \text{(a)} \\ U_1^{\mathrm{o}h}\big|_{N=0} = -\, u_1^{\mathrm{i}h}\big|_{N=0}. & \text{(b)} \end{cases} \tag{39}$$

Here, operators L_0 and L_1^h are introduced in (36). We are seeking outgoing solutions of (38) and (39), i.e., solutions which satisfy the limiting absorption principle.[3] We are not interested in the function U_1^{o}, because it does not describe the effect of a jump of curvature. This function solves the problem obtained from (39) via replacement L_1^h and $u_1^{\mathrm{i}h}$ by L_1 and u_1^{i}, respectively. It can be shown that U_1^{o} matches with $w_1^{\mathrm{r}} + v_1^{\mathrm{r}}$ (see (32) and (33)), which analysis we omit.

4. Investigation of term linear in *h*

Using formulas (30), we immediately find the solution of boundary value problem (38)

$$U_0^{\mathrm{o}} = -e^{i\mathrm{P}^+(S,N)}, \tag{40}$$

which is the leading-order term of reflected wave w_0^{r} (see (32)). With the help of (30), (39) and (40) we come up with the boundary value problem for $U_1^{\mathrm{o}h}$:

$$\begin{cases} \left(\frac{\partial^2}{\partial S^2} + \frac{\partial^2}{\partial N^2} + 1\right) U_1^{\mathrm{o}h} = \left((i\sin\beta + 2\cos^2\beta\, N)H(S) - i\cos\beta\, N\delta(S)\right) e^{i\mathrm{P}^+(S,N)}, & \text{(a)} \\ U_1^{\mathrm{o}h}\big|_{N=0} = -\frac{i\sin\beta}{2} S_+^2\, e^{iS\cos\beta}. & \text{(b)} \end{cases} \tag{41}$$

We assume that $U_1^{\mathrm{o}h}$ satisfies the limiting absorption principle.

It is easily verified that $U_1^{\mathrm{o}h}$ can be presented in the form

$$U_1^{\mathrm{o}h} = W + V, \tag{42}$$

where functions W and V are solutions of the following boundary value problems:

$$\begin{cases} \left(\frac{\partial^2}{\partial S^2} + \frac{\partial^2}{\partial N^2} + 1\right) W = \left((i\sin\beta + 2\cos^2\beta\, N)H(S) - i\cos\beta\, N\delta(S)\right) e^{i\mathrm{P}^+(S,N)}, & \text{(a)} \\ W\big|_{N=0} = \frac{i\sin\beta}{2} S_+^2\, e^{iS\cos\beta}, & \text{(b)} \end{cases} \tag{43}$$

and

$$\begin{cases} \left(\frac{\partial^2}{\partial S^2} + \frac{\partial^2}{\partial N^2} + 1\right) V = 0, & \text{(a)} \\ V\big|_{N=0} = -i\sin\beta\, S_+^2\, e^{iS\cos\beta}. & \text{(b)} \end{cases} \tag{44}$$

The limiting absorption principle guarantees uniqueness of solutions of problems (43) and (44).

Decomposition (42) is convenient for the following reasons. First, the solution of the problem (43) is easily found in elementary functions, while the problem (44) is far more complicated. Second, W matches with reflected wave, whereas V describes diffracted wavefield.

[3] Replacing 1 with $1 + i\epsilon$ in the expression for L_0 (see (36)), we replace (39) by a problem having an unique solution decreasing at infinity. Passing to a limit as $\epsilon \to 0$ (see, e.g., [16]), we obtain unique solution of (39).

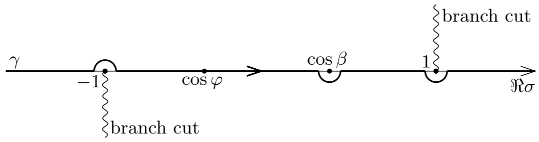

Fig. 4. Integration contour in (52).

4.1. Function W

Solution of (43) is very simple:

$$W = i\left(\frac{\sin\beta}{2}S_+^2 - \cos\beta\, NS_+\right)e^{i\mathrm{P}^+(S,N)}. \tag{45}$$

This expression coincides with w_1^{rh} (see (32)), whence the function W matches with the specularly reflected wave and is therefore of little interest.

4.2. Function V

We seek a solution of boundary value problem (44) in the form of Fourier integral

$$V(S,N) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mu(\sigma,N)\,e^{i\sigma S}d\sigma \tag{46}$$

with unknown density $\mu(\sigma,N)$. The Fourier transform $\mathcal{F}\left[S_+^2 e^{iS\cos\beta}\right](\sigma)$ of the function $S_+^2 e^{iS\cos\beta}$ is given by

$$\mathcal{F}\left[S_+^2 e^{iS\cos\beta}\right](\sigma) = \mathcal{F}\left[S_+^2\right](\sigma-\cos\beta) = 2i(\sigma-\cos\beta-i0)^{-3}, \tag{47}$$

see [17]. From (44) it ensues that $\mu(\sigma,N)$ satisfies the ordinary differential equation

$$\left(\frac{\partial^2}{\partial N^2}+1-\sigma^2\right)\mu = 0 \tag{48}$$

and the boundary condition

$$\mu|_{N=0} = 2\sin\beta\,(\sigma-\cos\beta-i0)^{-3}. \tag{49}$$

The problem (48), (49) has two linearly independent solutions:

$$\mu = 2\sin\beta\,(\sigma-\cos\beta-i0)^{-3}\,e^{\pm iN\sqrt{1-\sigma^2}}. \tag{50}$$

We take square root positive as $-1<\sigma<1$ with branch cuts shown in Fig. 4 by wavy lines. To satisfy the limiting absorption principle, we choose the function (50) with a plus sign in the exponent. We regularize the integral (46) by a deformation of contour of integration into the one shown in Fig. 4 ($\Re\sigma$ is the real part of σ). In a neighborhood of point O we introduce coordinates r and φ related to s and n by

$$s = r\cos\varphi,\quad n = r\sin\varphi,\quad -\pi<\varphi\le\pi, \tag{51}$$

and come up with:

$$V = \frac{\sin\beta}{\pi}\int_\gamma \frac{e^{ikr\left(\sqrt{1-\sigma^2}\sin\varphi+\sigma\cos\varphi\right)}}{(\sigma-\cos\beta)^3}\,d\sigma. \tag{52}$$

We assume that inequality (16) holds, whence $r\approx\rho=\sqrt{x^2+y^2}$. For

$$kr\to\infty, \tag{53}$$

standard asymptotic techniques (see, e.g., [18]) allows us to describe the function V by a sum of contributions of the pole $\sigma=\cos\beta$ and the critical point of phase $\sigma=\cos\varphi$. Hereafter we assume that the condition (53) holds true.

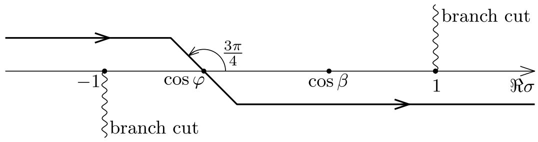

Fig. 5. Integration contour in (52) as $\cos\beta > \cos\varphi$.

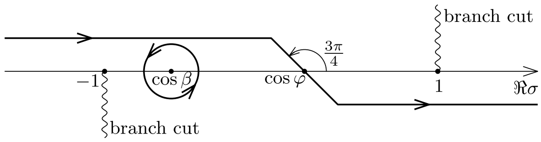

Fig. 6. Integration contour in (52) as $\cos\beta < \cos\varphi$.

4.2.1. Critical point of phase is not too close to pole

Let the observation point be positioned not too close to the limit ray and, accordingly, the critical point of phase be so far from the pole that asymptotics of (52) can be found by application of the standard method of steepest descent (see, e.g., [18]). As we will see later, this is guaranteed by condition

$$\sqrt{kr}|\varphi-\beta| \gg 1, \tag{54}$$

which is compatible with (16) and (17). The integration contour (see Fig. 4) is deformed into a one tangent to the steepest descent line (see, e.g., [18]) at the point $\sigma=\cos\varphi$, where its slope equals to $3\pi/4$. In the case where $\cos\beta > \cos\varphi$, the pole does not give a contribution to the integral (52), see Fig. 5. In the opposite case where $\cos\beta < \cos\varphi$, see Fig. 6, it does. We find the contribution of this third-order pole by the standard theory of residues (see, e.g., [19]).

Thus, asymptotics of the function hV/k which has a clear physical interpretation as an additive part of the wavefield, becomes

$$\begin{aligned}\frac{hV}{k} &= hH(\beta-\varphi)\left(\frac{n}{\sin^2\beta}-ik\frac{(s\sin\beta-n\cos\beta)^2}{\sin\beta}\right)e^{ik\mathrm{P}^{+}(s,n)}\\ &+\widetilde{A}(\varphi,k)\frac{e^{ikr}}{\sqrt{kr}}\left[1+O\left(\frac{1}{kr(\varphi-\beta)^2}\right)\right], \quad kr\to\infty,\end{aligned} \tag{55}$$

where

$$\widetilde{A}(\varphi,k)=\sqrt{\frac{2}{\pi}}\,\frac{h\sin\varphi\sin\beta}{k\,(\cos\varphi-\cos\beta)^3}\,e^{-i\frac{\pi}{4}}. \tag{56}$$

The first term on the right-hand side of (55) comes from the pole and matches with the function v_1^{rh} describing the specularly reflected wavefield (see (33)). The second comes from the critical point of phase and describes the diffracted wave u^{d}. Under the condition (54) the correction terms in (55) are small.

4.2.2. Matching (55) with (10)

Now we match the second term on the right-hand side of (55) with the cylindrical wave u^{d} (10) under condition (54). Relations (11), (24), (51) imply that $r=\rho+O\left(\varkappa\rho^2\right)$, whence $e^{ikr}=e^{ik\rho}\left(1+O\left(k\varkappa\rho^2\right)\right)$. Thus, the size of the boundary layer surrounding the point O is characterized by inequality (18).

Comparing (10) and (55) we obtain that $A(\phi,k)=\widetilde{A}(\phi,k)$ (see (56)) and arrive at the following expression for the diffracted wave u^{d}:

$$u^{\mathrm{d}}=\widetilde{A}(\phi,k)\frac{e^{ik\rho}}{\sqrt{k\rho}}\left(1+O\left(\frac{1}{k\rho(\phi-\beta)^2}\right)\right). \tag{57}$$

The correction term is small, provided that the condition

$$\sqrt{k\rho}|\phi-\beta| \gg 1, \tag{58}$$

(cf. (54)) is satisfied. The formula (57) perfectly agrees with results obtained earlier in [6,7] by Kirchhoff-type approaches.

4.2.3. Critical point of phase is close to pole

Consider the case where the condition (54) does not hold and derivation of the asymptotics of (52) requires a modification of the steepest-descent method. Let the observation point be positioned in the vicinity of the limit ray. Assume that

$$(kr)^{\frac{1}{3}}|\varphi-\beta| \ll 1. \tag{59}$$

The importance of the condition (59) is discussed in Appendix C. In a small neighborhood of the critical point of phase we introduce an integration variable $\sigma = \cos\varphi + \zeta\sin\varphi$, $|\zeta| \ll 1$. Following Erdélyi (see, e.g., [18]), we take a quadratic approximation for the phase in (52) and extend the integration contour from the neighborhood of the critical point of phase to infinity:

$$\frac{hV}{k} = \frac{h\sin\beta}{\pi k}\int_\gamma \frac{e^{ik\left(n\sqrt{1-\sigma^2}+s\sigma\right)}}{(\sigma-\cos\beta)^3}\,d\sigma \approx \frac{h}{\pi k\sin\beta}e^{ikr}\int_{\smile\rightarrow} \frac{e^{-\frac{ikr}{2}\zeta^2}}{(\zeta-(\varphi-\beta))^3}\,d\zeta. \tag{60}$$

Expression (60) can be easily written in terms of the parabolic cylinder function D_{-3} (see Appendix B):

$$\frac{hV}{k} \approx \sqrt{\frac{2}{\pi}}\frac{hr}{\sin\beta}\,e^{ikr-i\frac{z^2}{2}}D_{-3}\left(\sqrt{2}\,ze^{-i\frac{\pi}{4}}\right). \tag{61}$$

Here, we introduce the variable

$$z = \sqrt{\frac{kr}{2}}(\varphi-\beta), \tag{62}$$

which routinely arise in description of transition zones (typically, z^2 is the difference between phases of a plane and a cylindrical waves, see [9]). With the help of relation (see [20]) between D_{-3} and the Fresnel integral

$$F(Z) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{\pi}}\int_{-\infty}^{Z} e^{ip^2}\,dp, \tag{63}$$

formula (61) can be rewritten as

$$\frac{hV}{k} \approx \frac{ihr}{2\sin\beta}\,e^{ikr}\frac{d^2}{dz^2}\left[(1-F(z))\,e^{-iz^2}\right]. \tag{64}$$

4.2.4. Matching (61) with (55)

We will demonstrate that as both conditions (54) and (59) hold, expressions (55) and (61) asymptotically coincide. Indeed, using the asymptotics of the parabolic cylinder function D_{-3} (see (B.3) and (B.4)) and substituting the formula for z (62) in the expression (61) we obtain that as $z \to \pm\infty$

$$\begin{aligned}\frac{hv}{k} &\approx -\frac{hr}{\sin\beta}\left(2iz^2H(-z)e^{ikr-iz^2} + \frac{e^{ikr-i\frac{\pi}{4}}}{2\sqrt{\pi}\,z^3}\right)\\ &= -ikh\frac{r^2(\varphi-\beta)^2}{\sin\beta}H(\beta-\varphi)e^{ikr\left(1-\frac{(\varphi-\beta)^2}{2}\right)} + \sqrt{\frac{2}{\pi}}\frac{he^{-i\frac{\pi}{4}}}{k\sin\beta\,(\beta-\varphi)^3}\frac{e^{ikr}}{\sqrt{kr}}.\end{aligned} \tag{65}$$

It is easy to see that the leading-order terms of (55) and of (65) coincide.

5. Neighborhood of limit ray

In current section we introduce to our analysis a certain family of exact solutions of the Helmholtz equation, suitable for description the merging of diffracted and reflected waves.

5.1. Tsepelev's exact solutions of the Helmholtz equation

Let x' and y' be Cartesian coordinates with the origin at the point O and x'-axis directed along the specularly reflected ray (see Fig. 7). It is convenient to use classical parabolic coordinates ξ and η (see, e.g., [21]):

$$x' = \frac{1}{2}\left(\xi^2-\eta^2\right), \quad y' = \xi\eta. \tag{66}$$

In a small neighborhood of the limit ray, coordinate lines of ξ are approximately parallel and those of η are approximately orthogonal to it. The Helmholtz equation in parabolic coordinates ξ and η reads (see, e.g., [21])

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+k^2\left(\xi^2+\eta^2\right)\right)\mathfrak{U}(\xi,\eta) = 0. \tag{67}$$

Separation of variables allows a solution of (67):

$$\mathfrak{U}(\xi,\eta) = CD_{-\frac{1-q}{2}}\left(\sqrt{2k}\,\xi e^{-i\frac{\pi}{4}}\right)D_{-\frac{1+q}{2}}\left(\sqrt{2k}\,\eta e^{-i\frac{\pi}{4}}\right), \tag{68}$$

where D_ν is a parabolic cylinder function, C is arbitrary constant and q is separation parameter.

The family (68) was introduced by N.V. Tsepelev in the paper [22] aiming at description of wavefields near limit rays where waves of various nature merge.

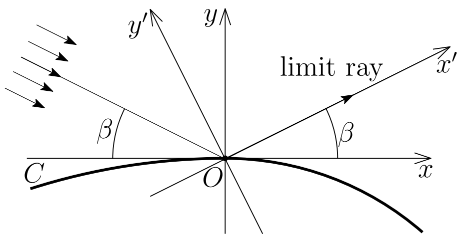

Fig. 7. System of coordinates x' and y'.

5.2. *Matching (68) with (61) within boundary layer surrounding O*

Now we match the solution of the Helmholtz equation (68) with expression (61) in the area described by (18) under an additional condition (59) which is satisfied in the vicinity of the limit ray. Here coordinates ξ and η (see (66)) are related to r and z (see (51) and (62)) by

$$\xi^2 = 2r(1 + o(1)), \quad z = \sqrt{k}\,\eta\,(1 + o(1))\,. \tag{69}$$

Comparing (61) and (68) we put $q = 5$ and (68) becomes

$$\mathfrak{U}(\xi,\eta) = CD_2\left(\sqrt{2k}\,\xi e^{-i\frac{\pi}{4}}\right) D_{-3}\left(\sqrt{2k}\,\eta e^{-i\frac{\pi}{4}}\right). \tag{70}$$

Substitution of the asymptotics of D_2 (see (B.3)) into (70) and using relations (69) give

$$\mathfrak{U} = -4iCkr\, e^{ikr - i\frac{z^2}{2}} D_{-3}\left(\sqrt{2}\,ze^{-i\frac{\pi}{4}}\right)(1 + o(1))\,, \quad kr \to \infty. \tag{71}$$

Comparison of (71) and (61) shows that asymptotic coincidence of these formulas requires that

$$C = \frac{ih}{2\sqrt{2\pi}\,k\sin\beta}. \tag{72}$$

5.3. *Matching (70) with outgoing wavefield beyond boundary layer surrounding O*

Now consider the area in the vicinity of the limit ray where condition (18) does not hold. Assume that $\sqrt{k}\,\xi \gg 1$ and $\sqrt{k}\,\eta \gg 1$, i.e., inequalities $k\rho \gg 1$ (cf. (53)) and (58) are satisfied. In a neighborhood of the limit ray where $|\phi - \beta| \ll 1$, we rewrite (70) in terms of polar coordinates ρ and ϕ (see (11), (66)) with the help of (B.3) and (B.4), and come up with

$$\begin{aligned}\mathfrak{U} = &\left(-ikh\frac{\rho^2(\phi-\beta)^2}{\sin\beta}H(\beta-\phi)e^{ik\mathrm{P}^+(x,y)} + \sqrt{\frac{2}{\pi}}\frac{he^{-i\frac{\pi}{4}}}{k\sin\beta\,(\beta-\phi)^3}\frac{e^{ik\rho}}{\sqrt{k\rho}}\right)\\ &\times\left[1 + O\left((\beta-\phi)^2\right) + O\left(\frac{1}{k\rho(\phi-\beta)^2}\right)\right].\end{aligned} \tag{73}$$

It can be observed that under conditions (16) and (17) the first term in round brackets matches with the reflected wave (see (13), (14), and (15)) and the second term with the diffracted wave (see (57) and (56)).

To summarize, the expression (70) with the constant C given by (72) describes the effect of jump of curvature on the outgoing wavefield in a narrow neighborhood (characterized by inequalities (16) and (17)) of the limit ray, where the phases of diffracted and reflected waves merge.

6. Conclusions

We have succeeded in application of a systematic boundary layer approach to high-frequency diffraction by a contour with a jump of curvature. We first described a wavefield in a neighborhood of the point of discontinuity, characterized by inequality (18). We presented an expression for a wavefield in the vicinity of the limit ray in terms of the parabolic cylinder function D_{-3}, which is valid under conditions (16) and (17). We developed techniques applicable to diffraction by a boundary or interface with curvature having higher order discontinuities, including the cases of point-source and tangent incidence.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

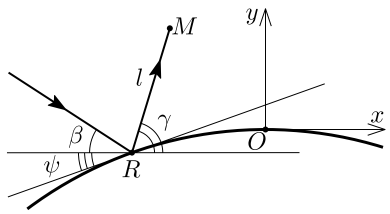

Fig. A.8. Reflection from a smooth part of contour.

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Appendix A. Derivation of expressions (13), (14), and (15) for reflected wave

Denote the distance between the observation point $M = (x, y)$ and the respective reflection point $R = (X, Y)$ by l. It is obvious that $l = (x - X)\cos\gamma + (y - Y)\sin\gamma$ (see Fig. A.8). The value of eikonal $\tau(X, Y)$ at the point R is related to the phase of incident wave (8) as follows: $\tau(X, Y) = \mathrm{P}^{-}(X, Y) = X\cos\beta - Y\sin\beta$ (see (27)). Thus, the value of eikonal at the point M is

$$\tau(x, y) = \tau(X, Y) + l = (x - X)\cos\gamma + (y - Y)\sin\gamma + X\cos\beta - Y\sin\beta. \tag{A.1}$$

The law of specular reflection reads

$$\beta + \psi = \gamma - \psi, \tag{A.2}$$

(see Fig. A.8). It is obvious from (A.2) that $\tan(\beta + 2\psi) = \tan\gamma$, which is equivalent to a helpful relation

$$\frac{\tan\beta + \tan(2\psi)}{1 - \tan\beta\tan(2\psi)} = \frac{y - Y}{x - X}. \tag{A.3}$$

Now assume that R is close to O. It follows from (20) and (22) that

$$X = s + O\left(\varkappa^2 s^3\right), \quad Y = -\frac{æ_0 s^2}{2} - \frac{h s_+^2}{2} + O\left(\varkappa^2 s^3\right), \tag{A.4}$$

where s is the arc length relevant to the point R. Employing the smallness of s and relations (A.4) and (22), we find from (A.3)

$$s = \frac{x\sin\beta - y\cos\beta}{\sin\beta}(1 + O(\varkappa\rho)). \tag{A.5}$$

With the help of (A.2) and (22) the eikonal (A.1) can be rewritten as follows:

$$\begin{aligned}\tau(x, y) &= (x - X)\cos(\beta + 2\psi) + (y - Y)\sin(\beta + 2\psi) + X\cos\beta - Y\sin\beta \\ &= x\cos\beta + y\sin\beta - 2\left(æ_0 s + h s_+\right)(y\cos\beta - x\sin\beta) - \left(æ_0 s^2 + h s_+^2\right)\sin\beta + O\left(\varkappa^2\rho^3\right).\end{aligned} \tag{A.6}$$

Finally, applying (A.5) gives

$$\tau(x, y) = x\cos\beta + y\sin\beta + (æ_0 + hH(\beta - \phi))\,\frac{(y\cos\beta - x\sin\beta)^2}{\sin\beta} + O\left(\varkappa^2\rho^3\right). \tag{A.7}$$

Expansion of the exponent in (12) up to quadratic terms and the amplitude up to linear terms allows formulas (13), (14), and (15).

Appendix B. Parabolic cylinder functions

Parabolic cylinder functions $D_\nu(\zeta)$ are defined as solutions of differential equation

$$\frac{d^2 D_\nu}{dZ^2} + \left(\nu + \frac{1}{2} - \frac{Z^2}{4}\right) D_\nu = 0 \tag{B.1}$$

satisfying conditions $D_\nu(0) = \sqrt{\pi}\,\frac{2^{\frac{\nu}{2}}}{\Gamma\left(\frac{1-\nu}{2}\right)}$, $D'_\nu(0) = -\sqrt{\pi}\,\frac{2^{\frac{\nu+1}{2}}}{\Gamma\left(\frac{-\nu}{2}\right)}$ (see, e.g. [20]). For integer values of ν function $D_\nu(Z)$ has the following integral representation (see, e.g., [23])

$$D_\nu(Z) = \frac{e^{i(\nu+1)\pi}}{\sqrt{2\pi}i} e^{\frac{Z^2}{4}} \int_{-\Theta+i\infty}^{-\Theta-i\infty} e^{Zp+\frac{1}{2}p^2} p^\nu \, dp, \tag{B.2}$$

where $\Theta > 0$. For large values of $|Z|$ and a fixed value of ν asymptotic expansion of D_ν is

$$D_\nu(Z) = Z^\nu e^{-\frac{Z^2}{4}} \left(1 + O\left(\frac{1}{|Z|^2}\right)\right), \tag{B.3}$$

as $-3\pi/4 < \arg Z < 3\pi/4$, and

$$D_\nu(Z) = \left(Z^\nu e^{-\frac{Z^2}{4}} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\nu\pi} Z^{-\nu-1} e^{\frac{Z^2}{4}}\right)\left(1 + O\left(\frac{1}{|Z|^2}\right)\right), \tag{B.4}$$

as $\pi/4 < \arg Z < 5\pi/4$ (see, e.g., [20]).

Appendix C. Correction terms in (61)

We are going to consider the calculation of Section 4.2.3 in more detail. In formula (60) we substitute the expansion of phase

$$ikr\left(\sqrt{1-\sigma^2}\sin\varphi + \sigma\cos\varphi\right) = ikr\left(1 - \frac{1}{2}\zeta^2\right) + O\left(kr\zeta^3\right) \tag{C.1}$$

and a relation $\sin\varphi = \sin\beta(1 + O(\varphi - \beta))$:

$$\frac{hv}{k} = \frac{h}{\pi k \sin\beta} e^{ikr} \int_{\smile\!\to} \frac{\left(1 + O(\varphi-\beta) + O\left(kr\zeta^3\right)\right) e^{-\frac{ikr}{2}\zeta^2}}{(\zeta - (\varphi-\beta))^3} \, d\zeta. \tag{C.2}$$

Remind that the condition (53) is assumed. Passing to integration variable $\tilde{\zeta} = \sqrt{kr}(\zeta - (\varphi-\beta))$ gives

$$\frac{hv}{k} = \frac{hr}{\pi\sin\beta} e^{ikr - ikr\frac{(\varphi-\beta)^2}{2}} \int_{\smile\!\to} \frac{e^{-i\sqrt{kr}(\varphi-\beta)\tilde{\zeta} - \frac{i}{2}\tilde{\zeta}^2}}{\tilde{\zeta}^3} \left(1 + O(\varphi-\beta) + O\left(\frac{(\tilde{\zeta} + \sqrt{kr}(\varphi-\beta))^3}{\sqrt{kr}}\right)\right) d\tilde{\zeta}. \tag{C.3}$$

Introduce a notation for small parameter $\vartheta = (kr)^{\frac{1}{3}}(\varphi-\beta)$ (see (59)). In terms of ϑ and kr we obtain

$$\begin{aligned}
&\frac{1}{\tilde{\zeta}^3}\left(1 + O(\varphi-\beta) + O\left(\frac{(\tilde{\zeta} + \sqrt{kr}(\varphi-\beta))^3}{\sqrt{kr}}\right)\right) \\
&= \frac{1}{\tilde{\zeta}^3}\left(1 + O\left((kr)^{-\frac{1}{3}}\vartheta\right) + O\left(\frac{(\tilde{\zeta} + (kr)^{\frac{1}{6}}\vartheta)^3}{\sqrt{kr}}\right)\right) \\
&= \frac{1}{\tilde{\zeta}^3} + O\left(\vartheta^3\right)\frac{1}{\tilde{\zeta}^3} + O\left((kr)^{-\frac{1}{3}}\vartheta\right)\left(\frac{1}{\tilde{\zeta}^3} + \frac{1}{\tilde{\zeta}}\right) + O\left((kr)^{-\frac{1}{6}}\vartheta^2\right)\frac{1}{\tilde{\zeta}^2} + O\left(\frac{1}{\sqrt{kr}}\right).
\end{aligned} \tag{C.4}$$

The integral in (C.3) is a sum of terms of orders $O(1)$, $O\left(\vartheta^3\right)$, $O\left((kr)^{-\frac{1}{6}}\vartheta^2\right)$, $O\left((kr)^{-\frac{1}{3}}\vartheta\right)$, and $O\left((kr)^{-\frac{1}{2}}\right)$, respectively. In the area where inequality (59) requiring that $|\vartheta| \ll 1$ holds, the relation (C.3) takes the form

$$\frac{hv}{k} = \sqrt{\frac{2}{\pi}}\frac{hr}{\sin\beta} e^{ikr - i\frac{z^2}{2}} D_{-3}\left(\sqrt{2}ze^{-i\frac{\pi}{4}}\right)\left(1 + O\left(\vartheta^3\right) + O\left(\frac{1}{\sqrt{kr}}\right)\right), \tag{C.5}$$

see (B.2), with z defined in (62).

We have shown that the correction terms in (C.5) are small under conditions (53) and (59).

References

[1] J.B. Keller, Geometrical theory of diffraction, J. Opt. Soc. Amer. 52 (2) (1962) 116–130, http://dx.doi.org/10.1364/JOSA.52.000116.

[2] A.V. Popov, Backscattering from a line of jump of curvature, in: Trudy V Vses. Sympos. Diffr. Raspr. Voln, Nauka, Leningrad, 1971, pp. 171–175.

[3] N.Ya. Kirpichnikova, V.B. Philippov, Diffraction of the whispering gallery waves near the line of the curvature jump, J. Math. Sci. 96 (4) (1999) 3342–3350, http://dx.doi.org/10.1007/BF02172810.

[4] N.Ya. Kirpichnikova, V.B. Philippov, A.S. Kirpichnikova, Diffraction of creeping waves by a line of jump of curvature (a three-dimensional acoustic medium), J. Math. Sci. 108 (5) (2002) 689–702, http://dx.doi.org/10.1023/A:1013247112086.

[5] A.S. Kirpichnikova, V.B. Philippov, Diffraction by a line of curvature jump (a special case), IEEE Trans. Antennas and Propagation 49 (12) (2001) 1618–1623, http://dx.doi.org/10.1109/8.982437.

[6] L. Kaminetzky, J.B. Keller, Diffraction coefficients for higher order edges and vertices, SIAM J. Appl. Math. 22 (1) (1972) 109–134, http://dx.doi.org/10.1137/0122013.
[7] A.F. Filippov, Reflection of a wave from a boundary composed of arcs of variable curvature, J. Appl. Math. Mech. (PMM) 34 (6) (1970) 1014–1023.
[8] Z.M. Rogoff, A.P. Kiselev, Diffraction at jump of curvature on an impedance boundary, Wave Motion 33 (2) (2001) 183–208, http://dx.doi.org/10.1016/S0165-2125(00)00041-X.
[9] V.A. Borovikov, B.E. Kinber, Geometrical Theory of Diffraction, Institute of Electrical Engineers, London, 1994, http://dx.doi.org/10.1049/PBEW037E.
[10] R.N. Buchal, J.B. Keller, Boundary layer problems in diffraction theory, Comm. Pure Appl. Math. 13 (1) (1960) 85–114, http://dx.doi.org/10.1002/cpa.3160130109.
[11] V.M. Babič, N.Ya. Kirpičnikova, The Boundary Layer Method in Diffraction Problems, Springer, Berlin, 1979.
[12] V.M. Babich, V.S. Buldyrev, Asymptotic Methods in Short-Wavelength Diffraction Theory, Alpha Science, Oxford, 2007.
[13] V.I. Smirnov, A Course of Higher Mathematics, Vol. 2, Pergamon Press, Oxford, 1964.
[14] G.S.S. Avila, J.B. Keller, The high-frequency asymptotic field of a point source in an inhomogeneous medium, Comm. Pure Appl. Math. 16 (4) (1963) 363–381, http://dx.doi.org/10.1002/cpa.3160160402.
[15] V.M. Babich, A.P. Kiselev, Elastic Waves: High-Frequency Theory, Chapman & Hall/CRC, London, 2018, http://dx.doi.org/10.1201/b21845.
[16] A.N. Tikhonov, A.A. Samarskii, Equations of Mathematical Physics, Pergamon Press, Oxford, 1963.
[17] I.M. Gelfand, G.E. Shilov, Generalized Functions: Properties and Operations, Vol. 1, Academic Press, New York & London, 1968.
[18] A. Erdélyi, Asymptotic Expansions, Dover Publications, New York, 1956.
[19] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, Cambridge University Press, Cambridge, 1915.
[20] H. Bateman, A. Erdélyi, Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York, 1953.
[21] P.M. Morse, H. Feshbach, Methods of Theoretical Physics, Vol. 1, McGraw-Hill, New York, 1953.
[22] N.V. Tsepelev, Some special solutions of the Helmholtz equation, J. Sov. Math. 11 (3) (1979) 497–501, http://dx.doi.org/10.1007/BF01088758.
[23] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, NBS, 1964.